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NOTES ON SYMMETRIC g-FUNCTIONS

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Abstract. A number of generalized metric spaces have been defined or characterized in terms of g-functions. Symmetric g-functions are discussed by C. Good, D. Jennings and A. M. Mohamad. In this paper, some questions about symmetric g-functions are answered, particularly it is shown that every sym-wg-space is expandable.

1. Introduction

A *g-function* on a topological space (X, τ) is a mapping $g : \omega \times X \to \tau$ such that $x \in g(n, x)$ for each $n \in \omega$ [5]. A number of generalized metric spaces have been defined or characterized in terms of g-functions, which generalized ball neighborhoods in metric spaces [4, 5, 6, 9]. Ball neighborhoods are symmetric. A g-function g on a space X is said to be *symmetric* if for each $n \in \omega$ and $x, y \in X, y \in g(n, x)$ whenever $x \in g(n, y)$. The symmetric g-functions are studied by C. Good, D. Jennings and A. M. Mohamad in [3]. It turns out that the majority of symmetric g-functions fall into one

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of four known classes of spaces, which are metrizable spaces, wM-spaces, o-semimetrizable spaces or β-spaces. The following questions are posed in [3].

QUESTION 1.1. Is every sym-wg-space discretely expandable?

QUESTION 1.2. Is every θ-refinable sym-wg-space paracompact?

QUESTION 1.3. Is every sym-wg-space with a G_δ-diagonal metrizable?

In this paper Questions 1.1 and 1.2 are affirmatively answered (Theorem 2.1 and Corollary 2.2), Question 1.3 is negatively answered (Example 2.4) and some results in [3] are improved.

Let $\mathcal{U}$ be a cover of a space X. For each $A \subset X$, put $\operatorname{st}(A,\mathcal{U}) = \cup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}$, and $\operatorname{st}^{k+1}(A,\mathcal{U}) = \operatorname{st}\big(\operatorname{st}^k(A,\mathcal{U}),\mathcal{U}\big)$ for all $k \in \omega$. It is easy to check that $\overline{\operatorname{st}(A,\mathcal{U})} \subset \operatorname{st}^2(A,\mathcal{U})$ if $\mathcal{U}$ is an open cover of X. We refer the reader to [1] or [2] for notations and terminology not explicitly given here; however we do not require paracompact spaces and θ-refinable spaces or sub-metacompact spaces to satisfy any separation axioms.

2. Main results

First recall some related definitions. A space (X,τ) is a *sym-wg-space* [3] if there is a symmetric g-function $g : \omega \times X \to \tau$ such that if $\{x, x_n\} \subset g(n, y_n)$ for all $n \in \omega$, then $\{x_n\}$ has a cluster point in X. A space X is said to be *expandable* [8] if for every locally finite collection $\{C_\alpha\}_{\alpha\in\Lambda}$ (without loss of generality closed sets) is expandable to a locally finite collection of open sets in X, i.e., there is a locally finite collection $\{U_\alpha\}_{\alpha\in\Lambda}$ of open sets of X such that $C_\alpha \subset U_\alpha$ for all $\alpha \in \Lambda$. X is said to be *discretely expandable* [10] if every discrete collection of sets of X is expandable to a locally finite collection of open sets.

It is obvious that every expandable space is discretely expandable. And it is shown that every normal, sym-wg, T_1-space is expandable [3, Theorem 18]. The main result in this paper is as follows.

THEOREM 2.1. *Every sym-wg-space is expandable.*

PROOF. Let (X,τ) be a sym-wg-space. There is a symmetric g-function $g : \omega \times X \to \tau$ such that if $\{x, x_n\} \subset g(n, y_n)$ for all $n \in \omega$, then $\{x_n\}$ has a cluster point in X. Without loss of generality, we can assume that each $g(n+1,x) \subset g(n,x)$. Put $\mathcal{U}_n = \big\{g(n,x) : x \in X\big\}$ for each $n \in \omega$. Then $\{\mathcal{U}_n\}$ is a sequence of open covers of X.

Step 1. For a sequence $\{x_n\}$ and x in X, if $x_n \in \operatorname{st}^2(x,\mathcal{U}_n)$ for all $n \in \omega$, then $\{x_n\}$ has a cluster point in X. There are $a_n, b_n, c_n \in X$ such that $x \in g(n,a_n), x_n \in g(n,b_n)$ and $c_n \in g(n,a_n) \cap g(n,b_n)$ by $x_n \in \operatorname{st}^2(x,\mathcal{U}_n)$. Then $\{c_n\}$ has a cluster point $c \in X$. There is a subsequence $\{c_{n_i}\}$ of

$\{c_n\}$ such that $c_{n_i} \in g(i,c)$ for all $i \in \omega$, thus $c \in g(i,c_{n_i})$ and $b_{n_i} \in g(n_i,c_{n_i}) \subset g(i,c_{n_i})$ for all $i \in \omega$ by the symmetry. Hence $\{b_{n_i}\}$ has a cluster point in X. By a similar method, $\{x_n\}$ has a cluster point in X.

Step 2. For a sequence $\{x_n\}$ and x in X, if $x_n \in \mathrm{st}^3(x,\mathcal{U}_n)$ for all $n \in \omega$, then $\{x_n\}$ has a cluster point in X. If not, then $\big\{\{x_n\}\big\}_{n\in\omega}$ is locally finite in X, thus $\big\{\mathrm{st}\,(x_n,\mathcal{U}_n)\big\}_{n\in\omega}$ is locally finite in X. Otherwise, there is $z \in X$ such that for each $n \in \omega$, $\mathrm{st}\,(z,\mathcal{U}_n) \cap \mathrm{st}\,\big(x_{i(n)},\mathcal{U}_{i(n)}\big) \neq \emptyset$ for some $i(n) \geqq n$. Then $x_{i(n)} \in \mathrm{st}\,\big(\mathrm{st}\,(z,\mathcal{U}_n),\mathcal{U}_{i(n)}\big) \subset \mathrm{st}^2(z,\mathcal{U}_n)$, and $\big\{x_{i(n)}\big\}$ has a cluster point in X by Step 1, a contradiction.

Now, $\mathrm{st}^2(x,\mathcal{U}_n) \cap \mathrm{st}\,(x_n,\mathcal{U}_n) \neq \emptyset$ by $x_n \in \mathrm{st}^3(x,\mathcal{U}_n)$, thus take $y_n \in \mathrm{st}^2(x,\mathcal{U}_n) \cap \mathrm{st}\,(x_n,\mathcal{U}_n)$ for all $n \in \omega$. Then $\{y_n\}$ has a cluster point in X by Step 1, a contradiction because $\big\{\mathrm{st}\,(x_n,\mathcal{U}_n)\big\}_{n\in\omega}$ is locally finite in X.

Step 3. If $\{B_n\}_{n\in\omega}$ is an increasing open cover of X, there is a locally finite open cover $\{G_n\}_{n\in\omega}$ of X with each $\overline{G}_n \subset B_n$. Put $F_n = X \setminus \mathrm{st}^2(X \setminus B_n,\mathcal{U}_n)$ for all $n \in \omega$. Then $\{F_n\}_{n\in\omega}$ is a closed cover of X. In fact, if $z \in X \setminus \bigcup_{n\in\omega} F_n = \bigcap_{n\in\omega} \mathrm{st}^2(X \setminus B_n,\mathcal{U}_n)$, then $\mathrm{st}^2(z,\mathcal{U}_n) \cap (X \setminus B_n) \neq \emptyset$, and take $z_n \in \mathrm{st}^2(z,\mathcal{U}_n) \cap (X \setminus B_n)$ for all $n \in \omega$. Let c be a cluster point of $\{z_n\}$ in X by Step 1. Then $c \in \bigcap_{n\in\omega}(X \setminus B_n)$, a contradiction.

Let $H_n = X \setminus \overline{\mathrm{st}\,(X \setminus B_n,\mathcal{U}_n)}$, then $F_n \subset H_n \subset \overline{H_n} \subset B_n$ for all $n \in \omega$. Thus $\{H_n\}_{n\in\omega}$ is an open cover of X. Put $B'_n = \bigcup_{i\leqq n} H_i$ for each $n \in \omega$. Then $\{B'_n\}_{n\in\omega}$ is an increasing open cover of X. By a similar method, there is an open cover $\{H'_n\}_{n\in\omega}$ of X with each $\overline{H'_n} \subset B'_n$. For each $n \in \omega$, put $G_n = H_n \setminus \bigcup_{i<n} \overline{H'_i}$, then G_n is open and $\overline{G}_n \subset B_n$.

$\{G_n\}_{n\in\omega}$ is a locally finite cover of X. In fact, let $x \in X$, then $x \in H'_m$ for some $m \in \omega$. Thus $H'_m \cap G_n = \emptyset$ for each $n > m$, which implies that $\{G_n\}_{n\in\omega}$ is locally finite in X. On the other hand, $\bigcup_{i<n} \overline{H'_i} \subset \bigcup_{i<n} B'_i = \bigcup_{i<n} H_i$, thus $H_n \setminus \bigcup_{i<n} H_i \subset G_n$, which implies that $\{G_n\}_{n\in\omega}$ is a cover of X.

Step 4. X is expandable. Let $\{F_\lambda\}_{\lambda\in\Gamma}$ be a locally finite collection of closed subsets of X. For each $n \in \omega$, put $A_n = \big\{x \in X : \big\{\lambda \in \Gamma : \mathrm{st}^2(x,\mathcal{U}_n) \cap F_\lambda \neq \emptyset\big\}$ is finite$\big\}$, $B_n = \mathrm{int}(A_n)$, then $B_n \subset B_{n+1}$. We shall check that $\{B_n\}_{n\in\omega}$ is a cover of X. In fact, for each $z \in X$, there is $n \in \omega$ such that $\big\{\lambda \in \Gamma : \mathrm{st}^3(z,\mathcal{U}_n) \cap F_\lambda \neq \emptyset\big\}$ is finite by Step 2, then $\mathrm{st}\,(z,\mathcal{U}_n) \subset A_n$, thus $z \in B_n$. There is a locally finite open cover $\{G_n\}_{n\in\omega}$ of X with each $\overline{G}_n \subset B_n$ by Step 3. For each $\lambda \in \Gamma, n \in \omega$, put $G_{\lambda n} = \mathrm{st}\,(F_\lambda,\mathcal{U}_n) \cap G_n$, $H_\lambda = \bigcup_{n\in\omega} G_{\lambda n}$, then H_λ is open in X, and $F_\lambda \subset H_\lambda$. We shall show that $\{H_\lambda\}_{\lambda\in\Gamma}$ is locally finite in X. In fact, for each $x \in X$, denote $\{n \in \omega : x \in \overline{G}_n\} = \{n_i :$

$i \leqq k\}$ for some $k \in \omega$, and $m(x) = \max\{n_i : i \leqq k\}$. Put $V = \mathrm{st}\big(x, \mathcal{U}_{m(x)}\big) \setminus \cup\{\overline{G}_n : x \notin \overline{G}_n,\ n \in \omega\}$, then V is an open neighborhood of x in X. For each $i \leqq k$, $\big\{\lambda \in \Gamma : \mathrm{st}\big(x, \mathcal{U}_{m(x)}\big) \cap G_{\lambda n_i} \neq \emptyset\big\} \subset \big\{\lambda \in \Gamma : \mathrm{st}^2(x, \mathcal{U}_{n_i}) \cap F_\lambda \neq \emptyset\big\}$ is finite by $x \in B_{n_i}$, thus V only meets with finitely many elements of $\{H_\lambda\}_{\lambda \in \Gamma}$. Hence X is expandable. □

COROLLARY 2.2. *Every θ-refinable sym-wg-space is paracompact.*

PROOF. Since every θ-refinable (= submetacompact) expandable space is paracompact [8, Theorem 2.11], every θ-refinable sym-wg-space is paracompact by Theorem 2.1. □

A space X is said to be have a G_δ*-diagonal* [4] if there is a sequence $\{\mathcal{U}_n\}$ of open covers of X such that $\{x\} = \bigcap_{n \in \omega} \mathrm{st}\,(x, \mathcal{U}_n)$ for each $x \in X$. A cover $\mathcal{U}$ of a space X is said to be T_1*-separating* [4] if whenever x and y are two points of X, then $x \in U \subset Y \setminus \{y\}$ for some $U \in \mathcal{U}$.

COROLLARY 2.3. *A space is metrizable if and only if it is a θ-refinable sym-wg, T_2-space with a G_δ-diagonal or a point-countable T_1-separating open cover.*

PROOF. The necessity is obvious. Let X be a θ-refinable sym-wg, T_2-space with a G_δ-diagonal or a point-countable T_1-separating open cover. Then X is a regular space by Corollary 2.2, thus it is metrizable by [3, Corollary 20]. □

A space X is said to be *developable* [6] if there exists a sequence $\{\mathcal{U}_n\}$ of open covers of X such that $\big\{\mathrm{st}\,(x, \mathcal{U}_n) : n \in \omega\big\}$ is a local base of x in X for each $x \in X$. A space X is developable if and only if it has a g-function g such that if $\{x, x_n\} \subset g(n, y_n)$ for all $n \in \omega$, then $\{x_n\}$ converges to x [6]. Every developable T_0-space has a G_δ-diagonal.

PROPOSITION 2.4. *There exists a G_δ-diagonal, sym-wg-space with a point-countable T_1-separating open cover, which is not metrizable.*

PROOF. Let $\mathbf{I} = [0,1]$ be the unit closed interval, and $X = \mathbf{I} \times \{0, 1, 1/2, 1/3, \dots\}$. X is endowed with the following topology τ. $\mathbf{I} \times \{1, 1/2, 1/3, \dots\}$ as an open subspace of X has the usual Euclidean topology; for each $a \in \mathbf{I}$, an element of a local base of $(a, 0)$ in X is of the form $\big(U \times \{0\}\big) \cup \big(\mathbf{I} \times \{1/k : k > n\}\big)$, where U is an open neighborhood of a in $\mathbf{I}$ with the usual Euclidean topology and $n \in \omega$.

It is easy to check that X is a T_1 and compact space with a countable base. Thus X is a θ-refinable sym-wg-space with a point-countable T_1-separating open cover. X is not metrizable because it is not a T_2-space.

A g-function $g : \omega \times X \to \tau$ on X is defined as follows. Let d be the usual Euclidean metric on the real line $\mathbf{R}$. For each $a \in \mathbf{I}$, and $n, m-1 \in \omega$, put

$$g\big(n, (a,0)\big) = \big(\big\{b \in \mathbf{I} : d(a,b) < 1/(n+1)\big\} \times \{0\}\big) \cup \big(\mathbf{I} \times \{1/k : k > n\}\big);$$

$$g\big(n,(a,1/m)\big)=\big\{b\in \mathbf{I}:\ d(a,b)<1/(n+1)\big\}\times\{1/m\}.$$

It is not difficult to verify that if $\{x,x_n\}\subset g(n,y_n)$ for all $n\in\omega$, then $\{x_n\}$ converges to x in X. Thus X is developable, and X has a G_δ-diagonal. The g-function g on X is not symmetric. In fact, there is no symmetric g-function on X satisfying the condition of developability because it has been shown that a T_0-space with the condition of developability is metrizable [3, Theorem 7]. □

QUESTION 2.5. Is every sym-wg, T_2-space with a G_δ-diagonal metrizable?

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