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Symmetric products and closed finite-to-one mappings ☆

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ABSTRACT

In this paper, we continue the study of the symmetric products of generalized metric spaces in [39]. We consider the topological properties $\mathscr{P}$ such that the n-fold symmetric product $\mathscr{F}_n(X)$ of a topological space X has the topological properties $\mathscr{P}$ if and only if the space X or the product X^n does for each or some $n \in \mathbb{N}$. Depending on the operations under closed subspaces, finite products and closed finite-to-one mappings, two general stability theorems are obtained on symmetric products. We can apply the methods to unify and simplify the proofs of some old results in the literature and obtain some new results on symmetric products, list or prove 68 topological properties which satisfy the general stability theorems, and give answers to Questions 3.6 and 3.35 in [39].

1. Introduction

Borsuk and Ulam [14] introduced the notion of a symmetric product of an arbitrary topological space. For a topological space X and each $n \in \mathbb{N}$ the n-fold symmetric product $\mathscr{F}_n(X)$ can be obtained as a quotient space of Cartesian product X^n. For the closed unit interval $\mathbb{I}$, Borsuk and Ulam proved that the n-fold

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symmetric product $\mathscr{F}_n(\mathbb{I})$ is homeomorphic to $\mathbb{I}^n$ for each $n \in \{1, 2, 3\}$, and $\mathscr{F}_n(\mathbb{I})$ is not homeomorphic with any subset of Euclidean space $\mathbb{R}^n$ for any $n \geq 4$, also that $\dim \mathscr{F}_n(\mathbb{I}) = n$ for each $n \in \mathbb{N}$. Bott [15] showed that $\mathscr{F}_3(\mathbb{S}^1)$ is homeomorphic to $\mathbb{S}^3$, where $\mathbb{S}^1$ and $\mathbb{S}^3$ are the unit circle and the three-sphere, respectively. Later, Ganea [33], Molski [76], Schori [85], Macías [65–68] et al. further investigated the symmetric products.

Recently, Good and Macías [39] studied the symmetric products of generalized metric spaces. They obtained some generalized metric properties $\mathscr{P}$ such that for a topological space X and each $n \in \mathbb{N}$, the space $\mathscr{F}_n(X)$ has the property $\mathscr{P}$ if and only if X does. It turns out that the behavior of the symmetric product topology mirrors the behavior of the usual product topology. Most interesting, their methods are constructive and do not relay on operations under products and closed mappings, which reveal the inner construction of the spaces X and $\mathscr{F}_n(X)$. They gave some examples of spaces X satisfying some properties, but $\mathscr{F}_2(X)$ does not. The following questions were posed.

Question 1.1. [39, Question 3.6] If X is a Lašnev space, then is $\mathscr{F}_n(X)$ a Lašnev space for some integer $n \geq 2$?

Question 1.2. [39, Question 3.35] Let X be a space and $n \in \mathbb{N}$. If $\mathscr{F}_n(X)$ is a Morita's P-space, then is X a Morita's P-space?

Metrizability and compactness are the heart and soul of general topology. Also for applications, these two concepts are the most important: metric notions are used almost everywhere in mathematical analysis, while compactness is used in many parts of analysis and also in mathematical logic. Besides metrizability and compactness, there are a few other concepts which are fundamental in general topology, for examples, generalized metric spaces and covering properties [23,42].

In this paper, we continue to consider the symmetric products of generalized metric spaces and covering properties. What is a *generalized metric space*? The term is meant for classes which are 'close' to metrizable spaces in some sense: they usually possess some of the useful properties of metric spaces, and some of the theory or techniques of metric spaces carries over to these wider classes. To be most useful, they should be 'stable' under certain topological operations, e.g., finite or countable products, closed subspaces, and perfect (i.e., closed, with compact point-inverses) mappings [42, p. 425]. A topological property is called a *covering property* if it can be characterized by every open cover of a space having a certain refinement, for examples, compactness, Lindelöfness, paracompactness, subparacompactness, etc. To be most useful, they should be 'stable' under certain topological operations, e.g., closed subspaces, and closed or perfect mappings, but they may not be finite or countable productive.

Although Good and Macías pointed out that [39, p. 94]: "Where ever possible we have proved our results directly rather than relying on preservation under products and closed maps", we still try to depend on the operations under closed subspaces, finite products and closed finite-to-one mappings, and apply the methods to unify and simplify the proofs of some old results in the literature and obtain some new results on symmetric products. In Section 3, we consider the topological properties $\mathscr{P}$ such that a topological space X has the property $\mathscr{P}$ if and only if $\mathscr{F}_n(X)$ does by a general stability theorem on the images of X^n for each $n \in \mathbb{N}$ (see Theorem 3.1), list or prove 43 properties which satisfy the general stability theorem, and give an affirmative answer to Question 1.2 in Theorem 3.10. In Section 4, we consider the topological properties $\mathscr{P}$ such that for a topological space X and each $n \in \mathbb{N}$, the product X^n has the property $\mathscr{P}$ if and only if $\mathscr{F}_n(X)$ does by another general stability theorem on the images of X^n and the inverse images of $\mathscr{F}_n(X)$ (see Theorem 4.1), list or prove 25 topological properties which satisfy the general stability theorem, and Question 1.1 is negatively answered in Example 4.15.

A lot of topological properties satisfy the general stability theorems, this paper lists just a part of them.

2. Preliminaries

All spaces are T_2 unless stated otherwise. The notation $\mathbb{N}$ denotes the set of all positive integers. The *mapping* stands for a continuous and surjective function and may be denoted by $f : X \to Y$. Readers may refer to [23,27,42] for notations and terminology not explicitly given here.

Let (X, τ) be a topological space, where τ is the topology for X. The following families of subsets of X are considered:

(1) $2^X = \{A \subseteq X : A \text{ is non-empty and compact}\}$;
(2) $\mathscr{F}(X) = \{A \in 2^X : A \text{ is finite}\}$;
(3) $\mathscr{F}_n(X) = \{A \in 2^X : |A| \leq n\}$, $n \in \mathbb{N}$.

It is obvious that $\mathscr{F}_n(X) \subseteq \mathscr{F}_{n+1}(X)$ for each $n \in \mathbb{N}$ and that $\mathscr{F}(X) = \bigcup_{n\in\mathbb{N}} \mathscr{F}_n(X)$.

We endow the set 2^X with the Vietoris topology τ_V, the base of which consists of all subsets of the following form:

$$\langle U_1, \ldots, U_k\rangle = \{A \in 2^X : A \subseteq \bigcup_{i\leq k} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \in \{1, \ldots, k\}\},$$

where each U_i is open in X for every $i \leq k$ and $k \in \mathbb{N}$.

Observe that the sets $\mathscr{F}_n(X)$ and $\mathscr{F}(X)$ are endowed with the subspace topology of 2^X. The space 2^X is called the *hyperspace of non-empty compact subsets* of X, the subspace $\mathscr{F}(X)$ is called the *hyperspace of finite subsets* of X and the subspace $\mathscr{F}_n(X)$ is called the *n-fold symmetric product* of X for each $n \in \mathbb{N}$.

The mapping $f_n : X^n \twoheadrightarrow \mathscr{F}_n(X)$ given by $f_n(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}$ will be used frequently in this paper.

Lemma 2.1. *Let $n \in \mathbb{N}$. Then*

(1) *$\mathscr{F}_n(X)$ is closed in $\mathscr{F}(X)$.*
(2) *$f_n : X^n \twoheadrightarrow \mathscr{F}_n(X)$ is a closed finite-to-one mapping [39].*

By Lemma 2.1, every $\mathscr{F}_m(X)$ is a closed subset of $\mathscr{F}_n(X)$ for each $m, n \in \mathbb{N}$, $m < n$, and $f_1 : X \twoheadrightarrow \mathscr{F}_1(X)$ is a homeomorphism.

Let X be a space. For every $P \subseteq X$, the set P is a *sequential neighborhood* of x in X if every sequence converging to x is eventually in P. The set P is a *sequentially open* subset of X if P is a sequential neighborhood of each point in P. A space X is said to be a *sequential space* [30] if each sequentially open subset is open in X. For each space (X, τ) the *sequential coreflection* [31] of (X, τ), denoted by (X, σ_τ) or σX, is given by $U \in \sigma_\tau$ if and only if U is sequentially open in (X, τ). As it is well known, σX is a sequential space [31, p. 52]; also, X and σX have the same convergent sequences [9, p. 678].

Let X be a space and $x \in X$. Suppose that $\mathscr{P}$ is a family of subsets in X. The family $\mathscr{P}$ is called a *network of x* [1] if $x \in \cap\mathscr{P}$, and $x \in U$ with U open in X, then $P \subseteq U$ for some $P \in \mathscr{P}$.

Definition 2.2. Let $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$ be a cover of a space X such that for each $x \in X$, (a) $\mathscr{P}_x$ is a network of x in X; (b) if $U, V \in \mathscr{P}_x$, then $W \subseteq U \cap V$ for some $W \in \mathscr{P}_x$.

(1) The family $\mathscr{P}$ is called an *sn-network* [59] for X if each element of $\mathscr{P}_x$ is a sequential neighborhood of x in X for each $x \in X$. A space X is called *snf-countable* [59] if X has an *sn*-network $\mathscr{P}$ such that each $\mathscr{P}_x$ is countable. A regular space X is called *sn-metrizable* [37] if X has a σ-locally finite *sn*-network.

(2) The family $\mathscr{P}$ is called an *so-network* [59] for X if each element of $\mathscr{P}_x$ is sequentially open in X for each $x \in X$. A space X is called *sof-countable* [59] if X has an *so*-network $\mathscr{P}$ such that each $\mathscr{P}_x$ is countable. A regular space X is called *so-metrizable* [35] if X has a σ-locally finite *so*-network.

(3) The family $\mathscr{P}$ is called a *weak base* [4] for X if, for every $A \subseteq X$, the set A is open in X whenever for each $x \in A$ there exists $P \in \mathscr{P}_x$ such that $P \subseteq A$. A space X is called *weakly first-countable* or *gf-countable* [4] if X has a weak base $\mathscr{P}$ such that each $\mathscr{P}_x$ is countable. A regular space X is called *g-metrizable* [87] if X has a σ-locally finite weak base.

Definition 2.3. Let $\mathscr{P} = \bigcup_{x \in X} \mathscr{P}_x$ be a cover of a space X, where each $x \in \cap \mathscr{P}_x$. The family $\mathscr{P}$ is called a *cs-network* [46] (resp. *cs*-network* [34]) for X, if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to $x \in U$ with U open in X, there exists $P \in \mathscr{P}_x$ such that $\{x_n\}_{n\in\mathbb{N}}$ (resp. some subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$) is eventually in P and $P \subseteq U$. A space X is called *csf-countable* [60] (resp. *cs*f-countable* [8][1]) if X has a cs-network (resp. cs^*-network) $\mathscr{P}$ such that each $\mathscr{P}_x$ is countable.

The following results hold for separation properties of hyperspaces [71, Theorem 4.9]: a space X is a T_2 (resp. regular, completely regular) space if and only if so is 2^X.

3. The images of X^n

In this section we discuss the topological properties $\mathscr{P}$ such that a space X has the property $\mathscr{P}$ if and only if $\mathscr{F}_n(X)$ does for each or some $n \in \mathbb{N}$. We obtain a general stability theorem as follows.

Theorem 3.1. *Suppose a topological property $\mathscr{P}$ satisfies the following:*
(1) *$\mathscr{P}$ is closed hereditary;*
(2) *$\mathscr{P}$ is finite productive; and*
(3) *$\mathscr{P}$ is preserved under closed finite-to-one mappings.*
Let X be a space and $n \in \mathbb{N}$. Then X has the property $\mathscr{P}$ if and only if $\mathscr{F}_n(X)$ does.

Proof. If a space X has the property $\mathscr{P}$, then the product X^n has the property $\mathscr{P}$ by condition (2). By Lemma 2.1, the mapping $f_n : X^n \twoheadrightarrow \mathscr{F}_n(X)$ is a closed finite-to-one mapping, then $\mathscr{F}_n(X)$ has the property $\mathscr{P}$ by condition (3).

Conversely, suppose $\mathscr{F}_n(X)$ has the property $\mathscr{P}$. By condition (1) and Lemma 2.1, X has the property $\mathscr{P}$. $\square$

As the applications of the general stability theorem, we will list or prove 43 topological properties which satisfy the conditions in Theorem 3.1, see Remarks 3.2 and 3.4, Corollary 3.9, and Theorems 3.10 and 3.16. The most important fact is to find the closed mapping properties on topological spaces, in which readers may refer to [19,56].

Remark 3.2. The following properties $\mathscr{P}$ satisfy the conditions of Theorem 3.1, thus a space X has the property $\mathscr{P}$ if and only if $\mathscr{F}_n(X)$ does for each or some $n \in \mathbb{N}$, which were proved by the constructive methods in [39].

α-spaces [38,96],[2] $\aleph_0$-spaces [73], cosmic spaces [73], developable spaces [18,38], first-countable spaces [95], γ-spaces [38,52], locally compact spaces [27], metric spaces [78], Moore spaces [18,27,38], M_2-spaces [13,24], Nagata spaces [12,24],[3] stratifiable spaces [12,24], σ-spaces [27,80,88],[4] regular spaces [27], and r-spaces [56].

[1] By [8, Proposition 2], a space X is csf-countable if and only if it is cs^*f-countable.
[2] In [38] an α-space is called a $\sigma^\sharp$-space and that, by [38, Lemma 4.1], the definition given by the authors is equivalent to the one given in [38].
[3] Let $f : X \to Y$ be a closed finite-to-one mapping. If X is a Nagata space, then X is first countable. By Corollary 3.13, Y is first countable. Hence Y is a Nagata space by [12].
[4] Let $f : X \to Y$ be a closed finite-to-one mapping. If X is a σ-space, then X is regular. By [27, Theorem 3.7.20], Y is regular. Then Y is a σ-space by [88].

We only check that r-spaces are preserved under closed finite-to-one mappings in the above properties,[5] others can be found in the relevant literature, or they are obvious. A topological space X is called an *r-space* [73] if for each $x \in X$ there exists a sequence $\{U_n(x)\}_{n\in\mathbb{N}}$ of open neighborhoods of x such that if $x_n \in U_n(x)$ for each $n \in \mathbb{N}$, then the set $\{x_n : n \in \mathbb{N}\}$ is contained in a compact set of X. The sequence $\{U_n(x)\}_{n\in\mathbb{N}}$ is called an *r-sequence* of x.

Lemma 3.3. *Let $f : X \to Y$ be a closed finite-to-one mapping. If X is an r-space, then so is Y.*

Proof. Let $y \in Y$ and $f^{-1}(y) = \{x_1, \dots, x_n\}$ for some $n \in \mathbb{N}$. Since X is an r-space, there is an r-sequence $\{U_m(x_i)\}_{m\in\mathbb{N}}$ of x_i for every $i \in \{1, \dots, n\}$. Obviously, $f^{-1}(y) \subseteq \bigcup_{i\le n} U_m(x_i)$ for each $m \in \mathbb{N}$. Since f is a closed mapping, there is an open neighborhood $V_m(y)$ of y such that $f^{-1}(V_m(y)) \subseteq \bigcup_{i\le n} U_m(x_i)$. Then $\{V_m(y)\}_{m\in\mathbb{N}}$ is an r-sequence of y. In fact, if a sequence $\{y_m\}_{m\in\mathbb{N}}$ of Y satisfies $y_m \in V_m(y)$ for each $m \in \mathbb{N}$. There exist $i_m \in \{1, \dots, n\}$ and $z_m \in f^{-1}(y_m) \cap U_m(x_{i_m})$. Therefore, the set $\{z_m : m \in \mathbb{N}\}$ is contained in a compact set of X. Thus, the set $\{y_m : m \in \mathbb{N}\} = f(\{z_m : m \in \mathbb{N}\})$ is contained in a compact set of Y. This completes the proof. □

Remark 3.4. The following properties $\mathscr{P}$ of topological spaces satisfy the conditions of Theorem 3.1, thus a space X has the property $\mathscr{P}$ if and only if $\mathscr{F}_n(X)$ does for each or some $n \in \mathbb{N}$, which does not list in [39].

$\aleph$-spaces [55,82], Čech-complete spaces [27],[6] hemi-compact spaces [56], k-metrizable spaces [7], k^*-metrizable spaces [7], k-semi-stratifiable spaces [34,64], k_ω-spaces [32], quasi-developable spaces [10,21], quasi-metrizable spaces [38,53], semi-metrizable spaces [38,95], semi-stratifiable spaces [25], sn-metrizable spaces [37], so-metrizable spaces [62], spaces of countable type [93], spaces of pointwise countable type [28; Appendix, Lemma 5.3], spaces with a $\delta\theta$-base [5,22], spaces with a point-countable base [5,28], spaces with a point-countable k-network [44], spaces with a σ-point-finite base [5,28], spaces with a uniform base [5,18, 94],[7] strict p-spaces [26,38], strong Σ-spaces [16,79], and strong $\Sigma^\sharp$-spaces [63].[8]

In the above properties we only check the following results: (1) hemi-compact spaces are preserved under perfect mappings[9]; and (2) k-metrizability satisfies the conditions of Theorem 3.1. A topological space X is called a *hemi-compact space* [27] if X has a countable cover of compact subsets such that every compact set of X is contained in one of them. A mapping $f : X \to Y$ is called a *proper mapping* [7, p. 477] if $f^{-1}(K)$ is a compact subset of X whenever K is a compact subset of Y. The proper images of metrizable spaces are called *k-metrizable spaces* [7, p. 484].

Lemma 3.5. *Let $f : X \to Y$ be a perfect mapping. If X is hemi-compact, then so is Y.*

Proof. Since X is hemi-compact, there is a countable cover $\{A_n\}_{n\in\mathbb{N}}$ of compact subsets in X such that each compact subset of X is contained in some A_n. Then $\{f(A_n)\}_{n\in\mathbb{N}}$ is a countable cover of compact subsets in Y. If C is a compact subset of Y, then $f^{-1}(C)$ is a compact subset of X, since f is a perfect mapping. Thus, $f^{-1}(C) \subseteq A_n$ for some $n \in \mathbb{N}$. That is $C \subseteq f(A_n)$, and Y is hemi-compact. □

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[5] Lemma 3.3 was announced in [56], its proof is new.

[6] Note that $f_n : X^n \twoheadrightarrow \mathscr{F}_n(X)$ is a closed finite-to-one mapping. If X is a Čech-complete space, so is X^n. According to [71, Theorem 4.9], $\mathscr{F}_n(X)$ is completely regular. Therefore, $\mathscr{F}_n(X)$ is Čech-complete by [27, Theorem 3.9.10].

[7] The papers [18] and [94] do not mention spaces with a uniform base. However, according to [27, Lemma 5.4.7], a space with a uniform base if and only if it is developable and metacompact. Let f be a perfect mapping from X onto Y. If X is a space with a uniform base, then X is developable and metacompact. By [18, p. 273, lines 2 through 7] and [94, Theorem 1, p. 175], Y is developable and metacompact. So Y is a space with a uniform base.

[8] It is easy to check that the countable product of strong $\Sigma^\sharp$-spaces is a strong $\Sigma^\sharp$-space, which proof is similar to strong Σ-spaces [79, Theorem 3.6]. The proof of the fact that being a strong $\Sigma^\sharp$-space is closed under countable unions of subspaces is similar to strong Σ-spaces [79, Theorem 3.2].

[9] Lemma 3.5 was announced in [56], its proof is new.

Lemma 3.6. *k-metrizability satisfies the conditions of Theorem 3.1.*

Proof. (1) k-metrizability is hereditary. Assume that Y is a k-metrizable space and Z is a subspace of Y. There are a metrizable space X and a proper mapping $f : X \to Y$. Let $S = f^{-1}(Z)$. Obviously, S is a metrizable space. If A is a compact subset of Z, then A is compact in Y. Therefore, $f^{-1}(A)$ is compact in X, and $(f|_S)^{-1}(A) = f^{-1}(A)$ is compact in S. Thus $f|_S : S \to Z$ is a proper mapping, and Z is a k-metrizable space.

(2) k-metrizability is countable productive. Suppose that $\{Y_n\}_{n\in\mathbb{N}}$ is a sequence of k-metrizable spaces. There are a sequence $\{X_n\}_{n\in\mathbb{N}}$ of metrizable spaces and a sequence $\{f_n : X_n \to Y_n\}_{n\in\mathbb{N}}$ of proper mappings. Define $f : \prod_{n\in\mathbb{N}} X_n \to \prod_{n\in\mathbb{N}} Y_n$ by $f((x_n)) = (f_n(x_n))$ for each $(x_n) \in \prod_{n\in\mathbb{N}} X_n$. It is easy to check that f is a proper mapping. Hence $\prod_{n\in\mathbb{N}} Y_n$ is a k-metrizable space.

(3) k-metrizability is preserved under closed finite-to-one mappings. Since the composition of two proper mappings is a proper mapping, k-metrizability is preserved under proper mappings. Clearly, every closed finite-to-one mapping is a proper mapping. Therefore, k-metrizability is preserved under closed finite-to-one mappings. □

Remark 3.7. The following properties $\mathscr{P}$ of topological spaces do not satisfy the conditions of Theorem 3.1, but it was proved in [39] that a space X has the property $\mathscr{P}$ if and only if $\mathscr{F}_n(X)$ does for each or some $n \in \mathbb{N}$.

Separable spaces [39, Theorem 3.10], spaces with a G_δ-diagonal (resp. G_δ^*-diagonal) [39, Theorem 3.21], and spaces with countable chain condition (MA+¬CH) [39, Corollary 5.5].[10]

Let $\mathscr{P}$ be a topological property which is closed under the countable unions of closed subspaces possessing the property $\mathscr{P}$. Hence, $\mathscr{F}(X)$ has the property $\mathscr{P}$ if each $\mathscr{F}_n(X)$ does by $\mathscr{F}(X) = \bigcup_{n\in\mathbb{N}} \mathscr{F}_n(X)$. The following is obtained.

Remark 3.8. The following properties $\mathscr{P}$ satisfy the condition: if a space X has the property $\mathscr{P}$, then so does $\mathscr{F}(X)$.[11]

Cosmic spaces [1,73], semi-stratifiable spaces [80], separable spaces, σ-spaces [80], strong Σ-spaces [79], and strong $\Sigma^\sharp$-spaces [74].

A space (X, τ) is called *submetrizable* [42] if there exists a topology τ' on X such that $\tau' \subseteq \tau$ and (X, τ') is metrizable. The submetrizability is not preserved by closed finite-to-one mappings [84, Example 1].

Corollary 3.9. *Let X be a space and $n \in \mathbb{N}$. Then X is a submetrizable space if and only if so is $\mathscr{F}_n(X)$.*

Proof. Since submetrizability is hereditary, it is only needed to prove the necessity. Let (X, τ) be a submetrizable space. There exists a topology τ' on X such that $\tau' \subseteq \tau$ and (X, τ') is metrizable. It is easy to see the Vietoris topologies $\tau'_V \subseteq \tau_V$ on 2^X. Put $\mathscr{T}'_n = \tau'_V|_{\mathscr{F}_n(X)}$ and $\mathscr{T}_n = \tau_V|_{\mathscr{F}_n(X)}$. Then the subspace topologies $\mathscr{T}'_n \subseteq \mathscr{T}_n$ on $\mathscr{F}_n(X)$, and $(\mathscr{F}_n(X), \mathscr{T}'_n)$ is metrizable by Theorem 3.1. Hence $\mathscr{F}_n(X)$ is submetrizable. □

The following result give a positive answer to Question 1.2. Recall the definition of Morita's P-spaces. A space X is called a *Morita's P-space* [77] if, for every collection $\{U(\alpha_1, \ldots, \alpha_n) : \alpha_1, \ldots, \alpha_n \in A; n \in \mathbb{N}\}$ of open subsets in X satisfying the following condition: $U(\alpha_1, \ldots, \alpha_n) \subseteq U(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ for every

[10] Separable property and spaces with countable chain condition are not closed hereditary [27, Example 2.3.12]; and spaces with a G_δ-diagonal (resp. G_δ^*-diagonal) are not preserved by a closed finite-to-one mapping [84, Example 1].

[11] Regularity does not satisfy the countable closed sum theorem [27, Example 1.5.6]. If X is a regular space, then 2^X is regular [71, Theorem 4.9], so is $\mathscr{F}(X)$.

$\alpha_1, \dots, \alpha_{n+1} \in A$ and $n \in \mathbb{N}$, there exists a collection $\{F(\alpha_1, \dots, \alpha_n) : \alpha_1, \dots, \alpha_n \in A; n \in \mathbb{N}\}$ of closed subsets in X satisfying the following conditions:

(1) $F(\alpha_1, \dots, \alpha_n) \subseteq U(\alpha_1, \dots, \alpha_n)$;

(2) $\bigcup_{n\in\mathbb{N}} F(\alpha_1, \dots, \alpha_n) = X$ if $\bigcup_{n\in\mathbb{N}} U(\alpha_1, \dots, \alpha_n) = X$ for each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$.

Theorem 3.10. *Let X be a space and $n \in \mathbb{N}$. Then X is a Morita's P-space if and only if so is $\mathscr{F}_n(X)$.*

Proof. Since the necessity was proved by [39, Theorem 3.34], it is enough to prove that every Morita's P-space is hereditary with respect to closed subspaces by Lemma 2.1.

Let X be a Morita's P-space, and Y be a closed subspace of X. Assume that a collection $\{U(\alpha_1, \dots, \alpha_n) : \alpha_1, \dots, \alpha_n \in A; n \in \mathbb{N}\}$ of open subsets in Y satisfies the following condition: $U(\alpha_1, \dots, \alpha_n) \subseteq U(\alpha_1, \dots, \alpha_n, \alpha_{n+1})$ for each $\alpha_1, \dots, \alpha_{n+1} \in A$ and $n \in \mathbb{N}$. Put $H(\alpha_1, \dots, \alpha_n) = U(\alpha_1, \dots, \alpha_n) \cup (X - Y)$. Clearly, $H(\alpha_1, \dots, \alpha_n)$ is open in X and $H(\alpha_1, \dots, \alpha_n) \subseteq H(\alpha_1, \dots, \alpha_n, \alpha_{n+1})$. By the definition of Morita's P-spaces, there exists a collection $\{F(\alpha_1, \dots, \alpha_n) : \alpha_1, \dots, \alpha_n \in A; n \in \mathbb{N}\}$ of closed subsets in X satisfying the following conditions:

(1) $F(\alpha_1, \dots, \alpha_n) \subseteq H(\alpha_1, \dots, \alpha_n)$;

(2) $\bigcup_{n\in\mathbb{N}} F(\alpha_1, \dots, \alpha_n) = X$ if $\bigcup_{n\in\mathbb{N}} H(\alpha_1, \dots, \alpha_n) = X$ for each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$.

It is easy to check that the family $\{F(\alpha_1, \dots, \alpha_n) \cap Y : \alpha_1, \dots, \alpha_n \in A; n \in \mathbb{N}\}$ is desired by Morita's P-spaces. Thus Y is a Morita's P-space. This completes the proof. □

It is well known that every Morita's P-space is preserved by closed mappings [77, Theorem 3.3]. The authors do not know whether the property of Morita's P-spaces are finite productive.

Next, we will show that snf-countable (resp. sof-countable, csf-countable) spaces satisfy the conditions in Theorem 3.1, which are preserved by closed finite-to-one mappings specifically.

A mapping $f : X \to Y$ is called a *sequentially quotient mapping* [11, p. 174] if, whenever $f^{-1}(H)$ is sequentially open in X for each $H \subseteq Y$, the set H is sequentially open in Y.[12] It is well known that f is sequentially quotient if and only if for each sequence $\{y_n\}_{n\in\mathbb{N}}$ converging to a point $y \in Y$, there exists a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X such that each $x_i \in f^{-1}(y_{n_i})$ for some subsequence $\{y_{n_i}\}_{i\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$, and the sequence $\{x_i\}_{i\in\mathbb{N}}$ converges to some $x \in f^{-1}(y)$ [11, Theorem 4.5].

Lemma 3.11. *Let $f : X \to Y$ be a closed finite-to-one mapping. Then f is a sequentially quotient mapping.*

Proof. Assume that $\{y_n\}_{n\in\mathbb{N}}$ is a sequence of Y converging to a point $y \in Y$. Let $S = \{y_n : n \in \mathbb{N}\} \cup \{y\}$. Then $f^{-1}(S)$ is a compact countable subset of X, thus $f^{-1}(S)$ is a compact metric space [42, Theorems 2.13 and 4.6]. For every $n \in \mathbb{N}$, choose a point $x_n \in f^{-1}(y_n)$. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ in $f^{-1}(S)$ has a convergent subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ in X. By [11, Theorem 4.5], f is sequentially quotient. □

Lemma 3.12. *Let $f : X \to Y$ be a closed finite-to-one mapping. If X is snf-countable, then so is Y.*

Proof. Let $y \in Y$ and $f^{-1}(y) = \{x_1, \dots, x_n\}$ for some $n \in \mathbb{N}$. Since X is snf-countable, there is a decreasing sn-network $\{U_{i,j}\}_{j\in\mathbb{N}}$ of x_i for every $i \in \{1, \dots, n\}$. Put $V_j = \bigcup_{i\leq n} U_{i,j}$ for each $j \in \mathbb{N}$. Then the set $f(V_j)$ is a sequential neighborhood of y in Y. If not, there is a sequence $\{y_n\}_{n\in\mathbb{N}}$ of Y converging to y such that $y_n \notin f(V_j)$ for each $n \in \mathbb{N}$. By Lemma 3.11, f is a sequentially quotient mapping, there is a sequence $\{z_k\}_{k\in\mathbb{N}}$ in X converging to a point $z \in f^{-1}(y)$ such that $y_{n_k} = f(z_k)$ for each $k \in \mathbb{N}$. Thus, there exists $i_0 \in \{1, \dots, n\}$ with $z = x_{i_0}$. Since the set $U_{i_0,j}$ is a sequential neighborhood of x_{i_0}, there is $k_0 \in \mathbb{N}$ such that $z_k \in U_{i_0,j} \subseteq V_j$ for every $k > k_0$. That is $y_{n_k} = f(z_k) \in f(V_j)$ for every $k > k_0$, which is

[12] Let $f : X \to Y$ be a continuous mapping. If a subset H of Y is sequentially open, then $f^{-1}(H)$ is sequentially open in X.

a contradiction. Next, we will show that the family $\{f(V_j)\}_{j\in\mathbb{N}}$ is an *sn*-network of y in Y. Let $y \in W$ with W open in Y. Then $f^{-1}(y) = \{x_1, \dots, x_n\} \subseteq f^{-1}(W)$. For every $i \in \{1, \dots, n\}$, there exists $j_i \in \mathbb{N}$ such that $x_i \in U_{i,j_i} \subseteq f^{-1}(W)$. Put $j = \max\{j_i : i \in \{1, \dots, n\}\}$. Then $f^{-1}(y) \subseteq V_j \subseteq f^{-1}(W)$, that is $y \in f(V_j) \subseteq W$. This completes the proof. □

A topological space X is called a *Fréchet space* [30] if, for any subset $A \subseteq X$ and $x \in \overline{A}$, there is a sequence in A converging to the point x in X.

Corollary 3.13. *Let $f : X \to Y$ be a closed finite-to-one mapping. If X is weakly first-countable (resp. first-countable), then so is Y.*

Proof. Since sequential (resp. Fréchet) spaces are preserved by closed mappings [30, Propositions 1.2 and 2.3], Y is a sequential (resp. Fréchet) space.[13] By Lemma 3.12, Y is an snf-countable space. It is well known that a space is weakly first-countable (resp. first-countable) if and only if it is sequential (resp. Fréchet) and snf-countable [Appendix, Lemma 5.1]. Hence Y is weakly first-countable (resp. first-countable). □

Lemma 3.14. *Let $f : X \to Y$ be a closed finite-to-one mapping. If X is sof-countable, then so is Y.*

Proof. Let $\sigma X, \sigma Y$ be the sequential coreflection of the spaces X, Y, respectively. Define $g : \sigma X \to \sigma Y$ by $g(x) = f(x)$ for each $x \in X$. Then g is finite-to-one.

Claim 1. σX is first-countable.

Let $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$ be an *so*-network for X. Since X is sof-countable, we can assume that the family $\mathscr{P}_x$ is a decreasing family of sequentially open subsets of X for each $x \in X$. Obviously, $\mathscr{P}$ is a family of open subsets of σX. We will show that the family $\mathscr{P}_x = \{P_{x,n}\}_{n\in\mathbb{N}}$ is a local base of x in σX. If not, there is an open neighborhood U of x in σX such that $P_{x,n} \setminus U \neq \emptyset$ for every $n \in \mathbb{N}$. Take $x_n \in P_{x,n} \setminus U$, $n \in \mathbb{N}$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x, thus U is not sequentially open in X, which is a contradiction. Hence, σX is first-countable.

Claim 2. g is a continuous and closed mapping.

Let F be closed in σY, i.e., the set F be sequentially closed in Y. Since f is continuous, $f^{-1}(F)$ is sequentially closed in X. Thus $f^{-1}(F)$ is closed in σX. Therefore, g is continuous.

Let A be a closed subset of σX. Suppose $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in $f(A)$ converging to a point y in Y. Choose $x_n \in A$ such that $y_n = f(x_n)$ for each $n \in \mathbb{N}$. By Lemma 3.11, it follows that there is a convergent subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ in X. Suppose that the sequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ converges to a point x in X. Since A is sequentially closed in X, the limit $x \in A$. Therefore, the sequence $\{f(x_{n_i})\} = \{y_{n_i}\}$ converges to the point y in Y. This implies $y = f(x) \in f(A)$, and $f(A)$ is sequentially closed in Y, i.e., $f(A)$ is closed in σY. Then $g : \sigma X \to \sigma Y$ is closed.

By Corollary 3.13 and Claims 1 and 2, the space σY is first-countable. Therefore, Y is sof-countable. □

Lemma 3.15. *Let $f : X \to Y$ be a closed finite-to-one mapping. If X is csf-countable, then so is Y.*

Proof. Suppose $y \in Y$ and $f^{-1}(y) = \{x_1, \dots, x_n\}$ for some $n \in \mathbb{N}$. Let $\{U_{i,j}\}_{j\in\mathbb{N}}$ be a *cs*-network of x_i for every $i \in \{1, \dots, n\}$. Put $\mathscr{P}_y = \{f(U_{i,j}) : i \in \{1, \dots, n\}, j \in \mathbb{N}\}$. We claim that the family $\mathscr{P}_y$ is a cs^*-network of y in Y. In fact, assume $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in Y converging to the point y, and V is a neighborhood of y in Y. By Lemma 3.11, f is a sequentially quotient mapping. Then there is a convergent sequence $\{z_k\}_{k\in\mathbb{N}}$ in X such that $\{f(z_k)\}_{k\in\mathbb{N}}$ is a subsequence of $\{y_n\}_{n\in\mathbb{N}}$. Suppose the sequence $\{z_k\}_{k\in\mathbb{N}}$

[13] In [30, p. 113, lines 25 through 27], the author mentioned that "It is easy to see that any open or closed map is pseudo-open and that each pseudo-open map is a quotient map". Therefore, sequential (resp. Fréchet) spaces are preserved by closed mappings [30, Propositions 1.2 and 2.3].

converges to $z \in f^{-1}(y)$. There is $i \in \{1, \ldots, n\}$ such that $z = x_i$. It follows from $z \in f^{-1}(V)$ that there is $j \in \mathbb{N}$ such that $U_{i,j} \subseteq f^{-1}(V)$ and the sequence $\{z_k\}_{k\in\mathbb{N}}$ is eventually in $U_{i,j}$. Thus, the sequence $\{y_n\}_{n\in\mathbb{N}}$ has a subsequence which is eventually in $f(U_{i,j})$ and $f(U_{i,j}) \subseteq V$. This shows that the family $\mathscr{P}_y$ is a cs^*-network of y in Y. Therefore, Y is cs^*f-countable. By [8, Proposition 2], Y is csf-countable. □

According to the definitions, it is easy to see that snf-countable spaces (resp. sof-countable spaces, csf-countable spaces) are closed hereditary and finite productive. Therefore, by Lemma 3.12 (resp. Lemmas 3.14 and 3.15) and Theorem 3.1, we have the following result.

Theorem 3.16. *Let X be a space and $n \in \mathbb{N}$. Then X is snf-countable (resp. sof-countable, csf-countable) if and only if $\mathscr{F}_n(X)$ is.*

A regular space with a σ-closure-preserved base is called an M_1-*space* [24]. In [39, Theorem 3.26], the authors proved the following result: Let X be a space and $n \in \mathbb{N}$. Then X is an M_1-space if and only if $\mathscr{F}_n(X)$ is. In the proof of [39, Theorem 3.26], the authors pointed out that each closed subset of an M_1-space is an M_1-space. Unfortunately, the proof maybe have a gap because we do not know whether each M_1-space is closed hereditary [43, Problem 1 and Theorem 1.1]. Therefore, we have the following question.

Question 3.17. Let X be a space. If $\mathscr{F}_n(X)$ is an M_1-space for some integer $n \geq 2$, then is X an M_1-space?

A completely regular space X is called a p-*space* [42, pp. 441–442] if there exists a sequence $\{\mathscr{U}_n\}_{n\in\mathbb{N}}$ of families of open subsets of the Čech-Stone compactification βX such that (i) each $\mathscr{U}_n$ covers X; (ii) for each $x \in X$, $\bigcap_{n\in\mathbb{N}} \mathrm{st}(x, \mathscr{U}_n) \subseteq X$.

Question 3.18. If X is a p-space, then is $\mathscr{F}_n(X)$ a p-space for some integer $n \geq 2$?

4. The inverse images of $\mathscr{F}_n(X)$

In this section, we will focus on the n-fold symmetric products of topological properties which are not finite productive. We discuss the topological property $\mathscr{P}$ such that the product X^n for a space X has the property $\mathscr{P}$ if and only if $\mathscr{F}_n(X)$ does for each $n \in \mathbb{N}$.

It is easy to prove the following general stability theorem by Lemma 2.1.

Theorem 4.1. *Suppose a topological property $\mathscr{P}$ satisfies the following:*

(1) *$\mathscr{P}$ is preserved under closed finite-to-one mappings; and*

(2) *$\mathscr{P}$ is an inverse invariant of closed finite-to-one mappings.*

Let X be a space and $n \in \mathbb{N}$. Then the product X^n has the property $\mathscr{P}$ if and only if $\mathscr{F}_n(X)$ does.

As the applications of the general stability theorem, we will list or prove 25 topological properties which satisfy the conditions in Theorem 4.1, see Remarks 4.2 and 4.3, Theorems 4.6, 4.8, 4.10, 4.12 and 4.14, and Corollary 4.11.

Remark 4.2. The conditions in Theorem 4.1 are satisfied by the following properties of topological spaces.

β-spaces [92; Appendix, Lemma 5.5], k-spaces [2,27], q-spaces [95; Appendix, Lemma 5.5], sequential spaces [31; Appendix, Lemma 5.4], Σ-spaces [79], $\Sigma^\sharp$-spaces [63,81], wM-spaces [49; Appendix, Lemma 5.5], and $w\sigma$-spaces [92; Appendix, Lemma 5.5].

Remark 4.3. The conditions in Theorem 4.1 are satisfied by the following covering properties of topological spaces.

Iso-compact spaces [6], Lindelöf spaces [27], mesocompact spaces [51,69],[14] metacompact spaces [40, 94], paracompact spaces [27,72], para-Lindelöf spaces [19,20], subparacompact spaces [16,17], θ-refinable spaces [19,50], and weakly θ-refinable spaces [23; Appendix, Lemma 5.6].

A space X is called a *strongly Fréchet* space [86] if, for every decreasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of subsets in X with $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, there is $x_n \in A_n$ for each $n \in \mathbb{N}$ such that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x.

Lemma 4.4. *Let $f : X \to Y$ be a closed finite-to-one mapping. If Y is a Fréchet (resp. strongly Fréchet) space, then so is Y.*

Proof. This lemma was announced in [56], its proof is new. We only prove the case of strongly Fréchet spaces. Let $\{A_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of subsets in X with $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$. Put $f^{-1}(f(x)) = \{x_1, \dots, x_n\}$ for some $n \in \mathbb{N}$ with $x_1 = x$. Since X is a T_2 space, there is an open neighborhood V of x such that $\overline{V} \cap \{x_2, \dots, x_n\} = \emptyset$. Therefore, $x \in V \cap \overline{A_n} \subseteq \overline{V \cap A_n}$ for each $n \in \mathbb{N}$. Thus, $f(x) \in f(\overline{V \cap A_n}) = \overline{f(V \cap A_n)}$. Since Y is a strongly Fréchet space, there is $z_n \in V \cap A_n$ for each $n \in \mathbb{N}$ such that the sequence $\{f(z_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$. By Lemma 3.11 and $\{A_n\}_{n\in\mathbb{N}}$ being a decreasing sequence of subsets, we can assume that the sequence $\{z_n\}_{n\in\mathbb{N}}$ is a convergent sequence in X, and its limit is in the set $\overline{V} \cap f^{-1}(f(x)) = \{x\}$, i.e., $\{z_n\}_{n\in\mathbb{N}}$ converges to x. Therefore, X is a strongly Fréchet space. □

Let (X, τ) be a space. A function $g : \mathbb{N} \times X \to \tau$ is called a *g-function* on X if, for every $x \in X$ and $n \in \mathbb{N}$, $x \in g(n + 1, x) \subseteq g(n, x)$. A space X is called a *$w\gamma$-space* [47] if, there is a g-function on X such that if $y_n \in g(n, p)$ and $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ has an accumulation point in X, and the g-function is called a *$w\gamma$-function.*

Lemma 4.5. *Let $f : X \to Y$ be a quasi-perfect mapping.*[15] *If X is a $w\gamma$-space, then so is Y.*

Proof. Since (X, τ_X) is a $w\gamma$-space, there is a $w\gamma$-function $g : \mathbb{N} \times X \to \tau_X$ on X.

Claim. If a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X has an accumulation point, then any sequence $\{z_n\}_{n\in\mathbb{N}}$ with $z_n \in g(n, x_n)$ has an accumulation point in X.

In fact, suppose that x is an accumulation point of the sequence $\{x_n\}_{n\in\mathbb{N}}$ in X. There is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_k} \in g(k, x)$ for each $k \in \mathbb{N}$. Therefore, $z_{n_k} \in g(n_k, x_{n_k}) \subseteq g(k, x_{n_k})$. Then the sequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ has an accumulation point in X.

Now, define a function $h : \mathbb{N} \times Y \to \tau_Y$ into the topology space (Y, τ_Y) by $h(n, y) = Y \setminus f(X \setminus \cup\{g(n, x) : f(x) = y\})$. Then h is a g-function on Y since f is closed. We will show that h is a $w\gamma$-function on Y. In fact, for each $q \in Y$, let $\{b_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences of Y such that $b_n \in h(n, q)$ and $y_n \in h(n, b_n)$ for each $n \in \mathbb{N}$. Since $f^{-1}(y_n) \subseteq f^{-1}(h(n, b_n)) \subseteq \bigcup\{g(n, x) : f(x) = b_n\}$, there exist $a_n \in f^{-1}(b_n)$ and $x_n \in f^{-1}(y_n) \cap g(n, a_n)$. And since $f^{-1}(b_n) \subseteq f^{-1}(h(n, q)) \subseteq \bigcup\{g(n, x) : f(x) = q\}$, there is $p_n \in f^{-1}(q)$ such that $a_n \in g(n, p_n)$. The sequence $\{p_n\}_{n\in\mathbb{N}}$ has an accumulation point, because $f^{-1}(q)$ is countably compact in X. By Claim, $a_n \in g(n, p_n)$ and $x_n \in g(n, a_n)$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ has an accumulation point in X. Therefore, $\{y_n\}_{n\in\mathbb{N}}$ has an accumulation point in Y by each $x_n \in f^{-1}(y_n)$. Thus, Y is a $w\gamma$-space. □

Since Fréchet spaces (resp. strongly Fréchet spaces) are preserved by pseudo-open mappings [30, Proposition 2.3] (resp. countably bi-quotient mappings [86, Proposition 3.4]) and $w\gamma$-spaces are an inverse invariant under quasi-perfect mappings [Appendix, Lemma 5.5], the following result is obvious by Lemma 4.4 (resp. Lemmas 4.4, 4.5) and Theorem 4.1.

[14] Kao and Wu in [51] corrected a gap in [69] and showed that the image of a mesocompact space under a prefect mapping is mesocompact.

[15] A closed mapping $f : X \to Y$ is *quasi-perfect* [19] if, for every $y \in Y$, $f^{-1}(y)$ is a countably compact subset in X.

Theorem 4.6. *Let X be a space and $n \in \mathbb{N}$. Then the product X^n is a Fréchet space (resp. strongly Fréchet space, $w\gamma$-space) if and only if $\mathscr{F}_n(X)$ is.*

A topological space X is called having *countable tightness* [75, Definition 8.2 and Proposition 8.5][16] if, for any subset $A \subseteq X$ and $x \in \overline{A}$, there is a countable subset $C \subseteq A$ such that $x \in \overline{C}$.

Lemma 4.7. *Let $f : X \to Y$ be a closed finite-to-one mapping. If Y has countable tightness, then X does.*

Proof. Let $A \subseteq X$ and $x \in \overline{A}$. Put $f^{-1}(f(x)) = \{x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$ with $x = x_1$. Since the space X is T_2, there are disjoint open subsets V_1, V_2 of X such that $x \in V_1$ and $\{x_2, \ldots, x_n\} \subseteq V_2$. Therefore, $x \in V_1 \cap \overline{A} \subseteq \overline{V_1 \cap A}$. Thus, $f(x) \in f(\overline{V_1 \cap A}) = \overline{f(V_1 \cap A)}$. Since the space Y has countable tightness, there is a countable subset $C \subseteq V_1 \cap A$ such that $f(x) \in \overline{f(C)} = f(\overline{C})$. This implies $f^{-1}(f(x)) \cap \overline{C} \neq \emptyset$. Since $f^{-1}(f(x)) \cap \overline{C} \subseteq (V_1 \cup V_2) \cap \overline{V_1} = V_1$, it follows that $f^{-1}(f(x)) \cap \overline{C} = \{x\}$, that is $x \in \overline{C}$. Then X has countable tightness. □

Since a space of countable tightness is preserved by quotient mappings [75, Lemma 8.4], the following result is obtained by Lemma 4.7 and Theorem 4.1.

Theorem 4.8. *Let X be a space and $n \in \mathbb{N}$. Then the product X^n has countable tightness if and only if $\mathscr{F}_n(X)$ does.*

A space X is called a k-*space* [27, p. 152] if, for every $A \subseteq X$, the set A is closed in X if and only if the intersection of A with any compact subspace K of the space X is relatively closed in K. Every sequential space is a k-space with countable tightness [75, p. 119 and Lemma 8.3].

Example 4.9. There is a Fréchet space X such that $\mathscr{F}_2(X)$ is neither a k-space nor of countable tightness.

Proof. Let S_{ω_1} be the quotient space obtained by identifying all the limit points of the topological sum of ω_1 many non-trivial convergent sequences. Then the space S_{ω_1} is a Fréchet space, but the product $(S_{\omega_1})^2$ is neither a k-space [41, Lemma 5] nor of countable tightness [45, p. 303]. By Theorems 4.1 and 4.8, $\mathscr{F}_2(S_{\omega_1})$ is neither a k-space nor of countable tightness. □

A function $d : X \times X \to [0, \infty)$ is called a *symmetric* on the set X if, for each $x, y \in X$, (i) $d(x, y) = 0$ if and only if $x = y$; and (ii) $d(x, y) = d(y, x)$. Let $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for every $x \in X$ and $\varepsilon > 0$. A space X is called *symmetrizable* [4] if, there is a symmetric d on the set X such that a subset $U \subseteq X$ is open if and only if for each $x \in U$, there exists $\varepsilon > 0$ with $B(x, \varepsilon) \subseteq U$. Every symmetrizable space is weakly first-countable [4, p. 129]. Symmetrizability is not an inverse invariant of closed finite-to-one mappings [70, Example 4.8].

Theorem 4.10. *Let X be a space and $n \in \mathbb{N}$. Then the following are equivalent:*
(1) *X is symmetrizable and X^n is a k-space;*
(2) *X^n is a symmetrizable space;*
(3) *$\mathscr{F}_n(X)$ is a symmetrizable space.*

Proof. (1) ⇔ (2) by [89, Theorem 4.2].

(2) ⇒ (3). Since symmetrizable spaces are preserved under closed finite-to-one mappings [89, p. 110], $\mathscr{F}_n(X)$ is a symmetrizable space by Lemma 2.1.

[16] *Countable tightness* is called *the property determined by countable subsets* in [75].

(3) ⇒ (1). If $\mathscr{F}_n(X)$ is a symmetrizable space, then $\mathscr{F}_n(X)$ is a sequential space [87, 1.4] and X is a symmetrizable space. Since sequentiality is an inverse invariant of closed finite-to-one mappings by [Appendix, Lemma 5.4], the product X^n is a sequential space by Lemma 2.1, and X^n is a k-space. □

Corollary 4.11. *Let X be a space and $n \in \mathbb{N}$. Then the following are equivalent:*

(1) *X is g-metrizable and X^n is a k-space;*
(2) *X^n is a g-metrizable space;*
(3) *$\mathscr{F}_n(X)$ is a g-metrizable space.*

Proof. (1) ⇔ (2) by [90, Theorem 2.9].

(2) ⇒ (3). Since g-metrizable spaces are preserved under closed finite-to-one mappings [Appendix, Lemma 5.2], $\mathscr{F}_n(X)$ is a g-metrizable space by Lemma 2.1.

(3) ⇒ (1). Suppose $\mathscr{F}_n(X)$ is a g-metrizable space. Obviously, X is g-metrizable. Since every g-metrizable space is symmetrizable [87, 1.8], X^n is a k-space by Theorem 4.10. □

Theorem 4.12. *Let X be a space and $n \in \mathbb{N}$. Then the following are equivalent:*

(1) *X is snf-countable and X^n is a sequential space;*
(2) *X^n is a weakly first-countable space;*
(3) *$\mathscr{F}_n(X)$ is a weakly first-countable space.*

Proof. (1) ⇒ (2). Since X is snf-countable, it is easy to see that the product X^n is snf-countable. Therefore, X^n is a weakly first-countable space by [Appendix, Lemma 5.1].

(2) ⇒ (3). Since the mapping $f_n : X^n \twoheadrightarrow \mathscr{F}_n(X)$ is a closed finite-to-one mapping and the product X^n is a weakly first-countable space, by Corollary 3.13, $\mathscr{F}_n(X)$ is a weakly first-countable space.

(3) ⇒ (1). If $\mathscr{F}_n(X)$ is a weakly first-countable space, then X is snf-countable and $\mathscr{F}_n(X)$ is sequential. Since the mapping $f_n : X^n \twoheadrightarrow \mathscr{F}_n(X)$ is a closed finite-to-one mapping, the product X^n is a sequential space by [Appendix, Lemma 5.4]. □

Example 4.13. There is a g-metrizable space X such that $\mathscr{F}_2(X)$ is not a k-space.

Proof. Let $Y = S_2 \times (\mathbb{P} \cup \{0\})$, where S_2 is the Arens' space [27, Example 1.6.19] and $\mathbb{P}$ is the set of irrational numbers. Then the space Y is not a k-space [63, Example 1.8.6, p. 44]. Put $X = S_2 \oplus (\mathbb{P} \cup \{0\})$. Then the space X is a g-metrizable space because the spaces S_2 and $(\mathbb{P} \cup \{0\})$ are g-metrizable spaces. Since Y is a closed subset of X^2 and the property of k-spaces is closed hereditary, we can conclude that the product X^2 is not a k-space. Therefore $\mathscr{F}_2(X)$ is not a k-space by Lemma 2.1 or Remark 4.2. □

Example 4.13 shows also that there is a g-metrizable space (resp. symmetrizable space, weakly first-countable space, sequential space, k-space) X such that $\mathscr{F}_2(X)$ is not a g-metrizable space (resp. symmetrizable space, weakly first-countable space, sequential space, k-space).

A topological space X is called a *Lašnev space* [19] if it is a closed image of a metric space.

Theorem 4.14. *The following are equivalent for a space X:*

(1) *$\mathscr{F}_n(X)$ is a metrizable space for each $n \in \mathbb{N}$;*
(2) *$\mathscr{F}_n(X)$ is a metrizable space for some integer $n \geq 2$;*
(3) *$\mathscr{F}_n(X)$ is a Lašnev space for some integer $n \geq 2$;*
(4) *X^2 is a Lašnev space;*
(5) *X is a metrizable space.*

Proof. Clearly, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). (4) $\Leftrightarrow$ (5) by [48, Theorem B], and (1) $\Leftrightarrow$ (5) by Theorem 3.1 or Remark 3.2.

To complete the proof, we will show that (3) $\Rightarrow$ (4). Suppose $\mathscr{F}_n(X)$ is a Lašnev space for some integer $n \geq 2$. By Lemma 2.1, $\mathscr{F}_2(X)$ is a closed subset of $\mathscr{F}_n(X)$, then $\mathscr{F}_2(X)$ is a Lašnev space. By [39, Theorem 3.9], X^2 is a Lašnev space. □

The following example gives a negative answer to Question 1.1.

Example 4.15. There is a Lašnev space X such that $\mathscr{F}_n(X)$ is not a Lašnev space for any integer $n \geq 2$.

Proof. Let S_ω be the quotient space obtained by identifying all the limit points of the topological sum of ω many non-trivial convergent sequences. Then the space S_ω is a non-metrizable Lašnev space [63, Example 1.8.7, p. 45]. Hence $\mathscr{F}_n(S_\omega)$ is not a Lašnev space for any integer $n \geq 2$ by Theorem 4.14. □

A space (X, τ) is called a *$w\Delta$-space* [63, Definition B.3.36] if, there is a g-function $g : \mathbb{N} \times X \to \tau$ on X such that if $\{x, x_n\} \subseteq g(n, y_n)$ for each $n \in \mathbb{N}$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ has an accumulation point in X.

Question 4.16. Let X be a space and an integer $n \geq 2$. If the product X^n is a $w\Delta$-space, then is $\mathscr{F}_n(X)$ a $w\Delta$-space?

Remark 4.17. If $\mathscr{F}_n(X)$ is a $w\Delta$-space, then so is X^n by [63, Proposition 3.6.15]

Remark 4.18. Recently, L.-X. Peng and Y. Sun in [83] considered symmetric products of generalized metric spaces. Their methods are also constructive and do not relay on operations under products and closed mappings. They proved that: Let X be a topological space and let $n \in \mathbb{N}$. If $\mathscr{P}$ satisfies one of the following properties, then X satisfies $\mathscr{P}$ if and only if $\mathscr{F}_n(X)$ satisfies $\mathscr{P}$.

(G) [83, Theorem 4]; open (G) [83, Theorem 5]; spaces with a point-countable base [83, Theorem 7]; second-countable spaces [83, Proposition 8]; spaces with a regular G_δ-diagonal [83, Theorem 10]; semi-stratifiable spaces [83, Theorem 12]; semi-metrizable spaces [83, Theorem 13]; k-semistratifiable spaces [83, Theorem 16]; scattered spaces [83, Theorem 17]; spaces with a point-countable cs-network [83, Theorem 18]; spaces in which each compact subspace is metrizable [83, Theorem 23].

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5. Appendix

In this Appendix, we list the definitions of some topological spaces which are not defined in the paper, and also prove some results from the references in Chinese for the convenience of the reader by reviewer's suggestions. For the convenience of the references, the definitions of the concepts may not be cited in the original literature.

5.1. Definitions and properties

(1) A space X is called a *regular space* if, for every $x \in X$ and every closed set $F \subseteq X$ such that $x \notin F$ there exist disjoint open sets U_1, U_2 such that $x \in U_1$ and $F \subseteq U_2$.

(2) A space X is called a *completely space* if, for every $x \in X$ and every closed set $F \subseteq X$ such that $x \notin F$ there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) \subseteq \{1\}$.

(3) A space X is called *first-countable* if, for every $x \in X$, there is a countable neighborhood base of x in X.

(4) A space X is called *separable* if X has a countable dense subset.

(5) A space X is called *compact* if every open cover of X has a finite subcover.

(6) A space X is called *locally compact* if, for every $x \in X$, there exists a neighborhood U of x such that $\overline{U}$ is compact in X.

(7) A space X is called a *k_ω-space* [32] if there is a countable cover $\{C_n\}_{n\in\mathbb{N}}$ of compact subsets of X such that $A \subseteq X$ is closed in X if and only if $A \cap C_n$ is relatively closed in C_n for every $n \in \mathbb{N}$.

(8) A completely space X is called *Čech-complete* [27, p. 196] if it is a G_δ-set in some Hausdorff compactification of it.

(9) A space X is called *Lindelöf* if every open cover of X has a countable subcover.

(10) A space X is called *para-Lindelöf* [23, p. 367] if every open cover of X has a locally countable open refinement.

(11) A space X is called *paracompact* [23, p. 351] if every open cover of X has a locally finite open refinement.

(12) A space X is called *subparacompact* [23, p. 360] if every open cover of X has a σ-discrete closed refinement.

(13) A space X is called *iso-compact* [6, p. 589] if every closed countably compact subset of X is compact.

(14) A space X is called *mesocompact* [69, Definition 1.1] if every open cover of X has a compact-finite open refinement.

(15) A space X is called *metacompact* [23, p. 362] if every open cover of X has a point-finite open refinement.

(16) A space X is called *weakly θ-refinable* [23, p. 362] if, for any open cover $\mathscr{U}$ of X there is an open refinement $\mathscr{G} = \bigcup_{n\in\mathbb{N}} \mathscr{G}_n$ such that if $x \in X$ there is some $n \in \mathbb{N}$ such that $1 \leq \mathrm{ord}(x, \mathscr{G}_n) < \omega$. If this condition is strengthened to require that each $\mathscr{G}_n$ also covers X, then X is said to be *θ-refinable* [23, p. 362].

(17) A space X is called of *countable type* [63, p. 85] if, for each compact subset F of X, there is a compact set K in X containing F such that K has a countable neighborhood base in X.

(18) A function $d : X \times X \to [0, \infty)$ is called a *distance* on X if d satisfies the following conditions for all $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) + d(y, z) \geq d(x, z)$.

Then (X, d) is called a *metric space*. For each $\varepsilon > 0$, let $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. The family $\{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a base for X.

(19) A space X is called *semi-metrizable* [42, p. 482] if there is a symmetric d on X such that for each $x \in X$, $\{B(x, \varepsilon) : \varepsilon > 0\}$ forms a neighborhood base of x.

(20) A function $d : X \times X \to [0, \infty)$ is called a *quasi-metric* [42, p. 488] on X if d satisfies the following conditions for all $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) + d(y, z) \geq d(x, z)$.

A space X is called *quasi-metrizable* [42, p. 488] if there is a quasi-metric d on X such that for each $x \in X$, $\{B(x, \varepsilon) : \varepsilon > 0\}$ forms a neighborhood base of x.

(21) Suppose $f : X \to Y$ is a mapping.

(i) f is called a *subproper mapping* [7, p. 477] if, there is a subset Z of X such that $f(Z) = Y$ and $\overline{Z \cap f^{-1}(K)}$ is a compact subset of X whenever K is a compact subset of Y.

(ii) The subproper images of metrizable spaces are called *k^*-metrizable spaces* [7, p. 484].

(22) A space X is said to have a *G_δ-diagonal* [42, p. 429] if $\Delta = \{(x,x) : x \in X\}$ is a G_δ-set in X^2.

(23) A space X is said to have a *G_δ^*-diagonal* [42, p. 432] if there exists a sequence $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ of open covers of X such that for each $x \in X$, $\{x\} = \bigcap_{n\in\mathbb{N}} \overline{\mathrm{st}(x,\mathscr{G}_n)}$.

(24) A sequence $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ of open covers of a space X is called a *development* [42, p. 426] for X if, for each $x \in X$ the family $\{\mathrm{st}(x,\mathscr{G}_n) : n \in \mathbb{N}\}$ is a neighborhood base of x. A *developable space* [42, p. 426] is a space with a development. A *Moore space* [42, p. 426] is a regular developable space.

(25) A space X is called *quasi-developable* [42, p. 479] if there exists a sequence $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ of families of open subsets of X such that for each $x \in X$, $\{\mathrm{st}(x,\mathscr{G}_n) : x \in \bigcup \mathscr{G}_n, n \in \mathbb{N}\}$ is a neighborhood base of x.

(26) A completely regular space X is called a *strict p-space* [42, pp. 441–442] if there exists a sequence $\{\mathscr{U}_n\}_{n\in\mathbb{N}}$ of families of open subsets of the Čech-Stone compactification βX such that

(i) each $\mathscr{U}_n$ covers X;

(ii) for each $x \in X$, $\bigcap_{n\in\mathbb{N}} \mathrm{st}(x,\mathscr{U}_n) = \bigcap_{n\in\mathbb{N}} \overline{\mathrm{st}(x,\mathscr{U}_n)} \subseteq X$.

(27) A space X is called a *stratifiable space* [42, p. 456] if, there is a function $H : \mathbb{N} \times \tau^c \to \tau$ satisfying the following conditions[17]:

(i) $n \in \mathbb{N}, F \in \tau^c \Rightarrow H(n,F) \supseteq F = \bigcap_{m\in\mathbb{N}} \overline{H(m,F)}$;

(ii) $L \subseteq F \Rightarrow H(n,L) \subseteq H(n,F)$.

(28) A space X is *semi-stratifiable* [63, p. 19] if there is a function $F : \mathbb{N} \times \tau \to \tau^c$ such that

(i) $U \in \tau \Rightarrow U = \bigcup_{n\in\mathbb{N}} F(n,U)$;

(ii) $V \subseteq U \Rightarrow F(n,V) \subseteq F(n,U)$.

Further assume X is a regular space and satisfies the following condition, then X is called a *k-semi-stratifiable space* [63, p. 19].

(iii) For every compact subset K of X with $K \subseteq U \in \tau$, there is $m \in \mathbb{N}$ such that $K \subseteq F(m,U)$.

(29) A collection $\mathscr{B}$ of subsets of X is a *quasi-base* for X if, whenever $x \in U$ with U open, then $x \in B^\circ \subseteq B \subseteq U$ for some $B \in \mathscr{B}$. An *M_2-space* [42, p. 465] is a regular space with a σ-closure-preserving quasi-base.

(30) A space X is called a *Nagata space* [24, Theorem 3.1] if it is a first-countable M_3-space.

(31) A family $\mathscr{P}$ of subsets of X is said to be *point-finite* [63, p. 13] (resp. *point-countable* [63, p. 11]) if each point of X only belongs to at most finite (resp. countably) many elements of $\mathscr{P}$. $\mathscr{P}$ is called a *k-network* [42, Definition 11.1, p. 493] for X if $K \subseteq U$ with K compact and U open in X, then $K \subseteq \cup\mathscr{P}' \subseteq U$ for some finite $\mathscr{P}' \subseteq \mathscr{P}$.

(i) A space X is called a *space with a point-countable base* [42, p. 472] (resp. *space with a point-countable k-network*) if X has a base (resp. k-network) which is point-countable.

(ii) A space X is called a *space with a σ-point-finite base* if X has a base which is the union of countable point-finite families.

(32) $\mathscr{P}$ is a *uniform base* [63, p. 134], if for each point $x \in X$ and each countably infinite subset $\mathscr{P}_1$ of $\{P \in \mathscr{P} : x \in P\}$, $\mathscr{P}_1$ is a neighborhood base at x.

(33) A regular space X is *cosmic* [63, p. 25] if X has a countable network.

(34) A regular space X is a *σ-space* [42, p. 446] if X has a σ-discrete (equivalently, σ-locally finite) network.

(35) A regular space X is an *$\aleph_0$-space* (resp. *$\aleph$-space*) [42, p. 493] if X has a countable (resp. σ-locally finite) k-network (see (31)).

(36) A space X is a (*strong*) *Σ-space* [42, p. 450] if there exist a σ-discrete collection $\mathscr{F}$ and a cover $\mathscr{C}$ of X by closed countably compact (compact) subsets, such that, whenever $C \in \mathscr{C}$ and $C \subseteq U$ with U open, then $C \subseteq F \subseteq U$ for some $F \in \mathscr{F}$.

[17] τ is the topology of X, and $\tau^c = \{X \setminus U : U \in \tau\}$.

(37) A space X is a (*strong*) $\Sigma^\sharp$*-space* [63, p. 160] if there exist a σ-closure-preserving collection $\mathscr{F}$ and a cover $\mathscr{C}$ of X by countably compact (compact) subsets, such that, whenever $C \in \mathscr{C}$ and $C \subseteq U$ with U open, then $C \subseteq F \subseteq U$ for some $F \in \mathscr{F}$.

(38) A collection $\mathscr{B} = \bigcup_{n\in\mathbb{N}} \mathscr{B}_n$ of open subsets of a space X is a $\delta\theta$*-base* [42, p. 477] if, whenever $x \in U$ with U open, there exist $n \in \mathbb{N}$ and $B \in \mathscr{B}_n$ such that

(i) $1 \leq \mathrm{ord}(x, \mathscr{B}_n) \leq \omega$;

(ii) $x \in B \subseteq U$.

(39) Suppose g is a g-function on a space X. Consider the following additional conditions:

(β) $p \in g(n, x_n) \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ has an accumulation point;

($w\sigma$) if $p \in g(n, y_n), y_n \in g(n, x_n) \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ has an accumulation point;

(q) $x_n \in g(n, p) \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ has an accumulation point;

(wM) $p \in g(n, z_n)$, $g(n, z_n) \cap g(n, y_n) \neq \emptyset$ and $y_n \in g(n, x_n) \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ has an accumulation point.

Spaces satisfying the above additional conditions are called β*-spaces* [42, p. 475], $w\sigma$*-spaces* [29], q*-spaces* [63, p. 35] and wM*-spaces* [63, p. 300] in turn.

(40) A space X is called an α*-space* [63, p. 27] if there is a g-function on X such that (i) $x \in g(n, x)$; (ii) if $y \in g(m, x)$, then $g(m, y) \subseteq g(m, x)$. In [63, p. 27] an α-space is called a $\sigma^\sharp$-space. Note that the definition of an α-space is also given in [39, p. 104].

(41) A space X is called a γ*-space* [42, p. 491] if there is a g-function on X such that (i) $\{g(n, x) : n \in \mathbb{N}\}$ is a neighborhood base of x; (ii) for every $n \in \mathbb{N}$ and $x \in X$ there exists $m \in \mathbb{N}$ such that $y \in g(m, x)$ implies $g(m, x) \subseteq g(n, x)$.

(42) A space X is called *of pointwise countable type* [3] if for each $x \in X$, there is a compact set K containing x such that K has a countable neighborhood base in X.

By the definitions, it is easy to check the following results:

(1) The following spaces are hereditary with respect to subspaces:

α-spaces, γ-spaces, developable spaces, first-countable spaces, Moore spaces, quasi-metric spaces, semi-metrizable spaces, sn-metrizable spaces, so-metrizable spaces, spaces with a point-countable base, spaces with a σ-point-finite base, spaces with a uniform base, spaces with a $\delta\theta$-base.

(2) The following spaces are hereditary with respect to closed subspaces:

Spaces of countable type, hemi-compact spaces, strict p-spaces.

(3) The following spaces are countably productive:

Spaces of countable type, spaces of pointwise countable type, spaces with a σ-point-finite base, sn-metrizable spaces, so-metrizable spaces.

5.2. Proofs

Our proofs may be different from the original proofs, because the purpose is to give concise proofs for the results.

Lemma 5.1. (1) *A space X is weakly first-countable if and only if X is sequential and snf-countable [59, Lemma 2.1].*

(2) *A space X is first-countable if and only if X is Fréchet and snf-countable [61].*

Proof. (1) If X is weakly first-countable, then X is sequential [87, 1.4]. Obviously, X is snf-countable.

Conversely, suppose X is sequential and snf-countable. Let $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$ be an sn-network for X, and $\mathscr{P}_x$ be countable for each $x \in X$. Then $\mathscr{P}$ is a weak base for X. In fact, if A is a subset of X such that for each $x \in A$ there is $P_x \in \mathscr{P}_x$ with $P_x \subseteq A$, then A is sequentially open in X. Since X is sequential, A is open. This shows that $\mathscr{P}$ is a weak base for X. Thus, X is weakly first-countable.

(2) If X is first-countable, then X is Fréchet [31, p. 113]. Obviously, X is snf-countable.

Conversely, suppose X is Fréchet and snf-countable. By (1), X is weakly first-countable. Since every weakly first-countable Fréchet space is first countable [87, Theorem 1.10], X is first countable. □

Lemma 5.2. *Let $f : X \to Y$ be a closed finite-to-one mapping. If X has one of the following properties, then so does Y:*

(1) g-metrizable spaces [57, Remark 3.1, p. 409]; (2) sn-metrizable spaces [37, Theorem 3.4]; (3) semi-metrizable spaces [95, Theorem 2(10)]; (4) q-spaces [95, Theorem 2(9)].

Proof. (1) In [91, Theorem 13], the author proved that the space Y is g-metrizable if and only if Y is weakly first-countable. Since X is weakly first-countable, Y is weakly first-countable by Corollary 3.13 in the body of the paper. Thus Y is g-metrizable.

(2) In [36, Theorem 1],[18] the author proved that a space is an sn-metrizable space if and only if it is an $\aleph$-space and snf-countable. If X is sn-metrizable, then X is an $\aleph$-space and snf-countable. By Lemma 3.12 in the body of the paper and [55, Theorem 2.2], Y is an $\aleph$-space and snf-countable. So Y is sn-metrizable.

(3) By [25, Corollary 1.4], a space is semi-metrizable if and only if it is semi-stratifiable and first-countable. If X is semi-metrizable, then X is semi-stratifiable and first-countable. Since f is a closed finite-to-one mapping, by [25, Theorem 3.1] and Corollary 3.13 in the body of the paper, Y is semi-stratifiable and first-countable. Hence Y is semi-metrizable.

(4) Since X is a q-space, there is a g-function g on X satisfying additional condition (q), see item (39) in 5.1. Suppose $y \in Y$ and $f^{-1}(y) = \{x_1, \dots, x_n\}$ for some $n \in \mathbb{N}$. Obviously, $f^{-1}(y) \subseteq \bigcup_{i \le n} g(m, x_i)$ for each $m \in \mathbb{N}$. Since f is a closed mapping, there is an open neighborhood $h(m, y)$ of y such that $f^{-1}(h(m, y)) \subseteq \bigcup_{i \le n} g(m, x_i)$. Define $k : \mathbb{N} \times Y \to \tau_Y$ by $k(j, y) = \bigcap_{m \le j} h(m, y)$, where τ_Y is the topology for Y. It is easy to see that k is a g-function on Y. We will show that k satisfies additional condition (q). In fact, let $\{y_j\}_{j \in \mathbb{N}}$ be a sequence of Y satisfying $y_j \in k(j, y)$ for each $j \in \mathbb{N}$. Then $f^{-1}(y_j) \subseteq \bigcup_{i \le n} g(j, x_i)$. Thus there are $i_m \le n$ and $z_j \in f^{-1}(y_j) \cap g(j, x_{i_m})$. We can assume that there is a strictly increasing sequence $\{m_l\}_{l \in \mathbb{N}} \subseteq \mathbb{N}$ such that $i_{m_l} = 1$ for each $l \in \mathbb{N}$. Therefore, $z_{j_l} \in g(j_l, x_1) \subseteq g(l, x_1)$. So $\{z_{j_l}\}_{l \in \mathbb{N}}$ has an accumulation point in X. It implies that $\{y_{j_l}\}_{l \in \mathbb{N}}$ has an accumulation point in Y. Then Y is a q-space. □

Lemma 5.3. *[58, Theorem] Let $f : X \to Y$ be a closed finite-to-one mapping. If X is a space of pointwise countable type, then so is Y.*

Proof. Assume $y \in Y$. There is a compact subset K of X containing $f^{-1}(y)$ such that K has an open neighborhood base $\{U_n\}_{n \in \mathbb{N}}$ in X. Put $V_0 = X$. Since f is closed, we can define, by induction, a sequence $\{V_n\}_{n \in \mathbb{N}}$ of open subsets of X such that for each $n \in \mathbb{N}$:

(1) $f^{-1}(y) \subseteq V_n \subseteq V_{n-1} \cap U_n$;

(2) $f^{-1}(f(V_n)) = V_n$;

(3) $\overline{V_n \cap K} \subseteq V_{n-1} \cap K$.

In fact, let $V_1 = f^{-1}(Y \setminus f(X \setminus U_1))$. Since f is closed, V_1 is open. It is easy to check that the set V_1 satisfies (1), (2) and (3). Assume that we have constructed $V_1, \dots, V_k$ satisfying (1), (2) and (3). Since $f^{-1}(y) \subseteq V_k \cap K$ and K is a normal subspace of X, there is an open subset W_k in K such that $f^{-1}(y) \subseteq W_k \subseteq \mathrm{cl}_K(W_k) \subseteq V_k \cap K$. Then there is an open subset G_k in X such that $W_k = G_k \cap K$. Thus, $\overline{G_k \cap K} = \overline{W_k} = \mathrm{cl}_K(W_k) \subseteq V_k \cap K$ and $f^{-1}(y) \subseteq G_k \cap U_{k+1} \cap V_k$. Put $V_{k+1} = f^{-1}(Y \setminus f(X \setminus G_k \cap U_{k+1} \cap V_k))$. Hence V_{k+1} is open in X and satisfies (1), (2) and (3). Therefore, the construction is completed.

Put $H = \bigcap_{n \in \mathbb{N}} V_n$. Then $f^{-1}(y) \subseteq H \subseteq f^{-1}(f(H))$ and $H \subseteq \bigcap_{n \in \mathbb{N}} U_n$. Since $\{U_n\}_{n \in \mathbb{N}}$ is an open neighborhood base of K in X, we have $K = \bigcap_{n \in \mathbb{N}} U_n$. Thus $H = \bigcap_{n \in \mathbb{N}} (V_n \cap K)$. By (3), $H = \bigcap_{n \in \mathbb{N}} \overline{V_n \cap K}$.

[18] In [36] an sn-metrizable space is called a regular space which has a σ-locally finite cs-network, and an snf-countable space is called a universally csf-countable space.

Therefore, H is compact. Suppose that O is an open neighborhood of H in X. By [27, Corollary 3.1.5], there is $n \in \mathbb{N}$ such that $\overline{V_n \cap K} \subseteq O$. Hence $H \subseteq V_n \cap K \subseteq O \cap K$. This shows that H has a countable neighborhood base in the compact subspace K. Since K has an open neighborhood base in X, by [27, Exercise 3.1.E(a)], H has a countable neighborhood base in X. It shows that Y is a space of pointwise countable type. □

Lemma 5.4. *[97, Theorem 2, p. 7] Let $f : X \to Y$ be a closed finite-to-one mapping. If Y is a sequential space, then so is X.*

Proof. Let σX be the sequential coreflection of the space X, $s : \sigma X \to X$ be the identity mapping. Obviously, s is continuous. Put $g = f \circ s : \sigma X \to Y$, then g is a finite-to-one continuous mapping. If F is a closed set of σX, $g(F)$ is a closed set of Y. Otherwise, $g(F) = f(F)$ is not sequentially closed in Y since Y is sequential. Thus, there are a sequence $S \subseteq F$ and $y \in Y$ such that $f(S)$ converges to y with $y \notin f(F)$. By Lemma 3.11 in the body of the paper, f is a sequentially quotient mapping, we can assume that S is a convergent sequence. Since F is sequentially closed in σX, $y \in F$, the limit point of S is contained in F. Then $y \in f(F)$, which is a contradiction. It implies that g is a closed mapping. Hence g is a perfect mapping. According to [27, Propersiton 3.7.5], s is a perfect mapping. So X is a sequential space. □

Lemma 5.5. *[54, Theorem 1] Let $f : X \to Y$ be a quasi-perfect mapping. If Y is a q-space (resp. β-space, $w\gamma$-space, $w\sigma$-space, wM-space), then so is X.*

Proof. We need only to consider the case of $w\gamma$-space, the others are similar.

Let Y be a $w\gamma$-space, there is a $w\gamma$-function g on Y. Define a function $h : \mathbb{N} \times X \to \tau_X$ by $h(n, x) = f^{-1}(g(n, f(x)))$, where τ_X is the topology for X. Then h is a g-function on X. Suppose that $b_n \in h(n, c)$, and $a_n \in h(n, b_n)$, then $f(b_n) \in g(n, f(x))$, and $f(a_n) \in g(n, f(b_n))$. Since Y is a $w\gamma$-space, the sequence $\{f(a_n)\}_{n\in\mathbb{N}}$ has an accumulation point. Thus, the family $\{\{f(a_n)\}\}_{n\in\mathbb{N}}$ is not locally finite. Since f is a quasi-perfect mapping, $\{\{a_n\}\}_{n\in\mathbb{N}}$ is not a locally finite family. Therefore, the sequence $\{a_n\}_{n\in\mathbb{N}}$ has an accumulation point. The proof is completed. □

A mapping $f : X \to Y$ is *Lindelöf* if, for every $y \in Y$, $f^{-1}(y)$ is a Lindelöf subset in X.

Lemma 5.6. *[98, Theorem 3] Let $f : X \to Y$ be a closed Lindelöf mapping. If Y is a weakly θ-refinable space, then so is X.*

Proof. Suppose $\mathscr{U}$ is an arbitrary open cover of X. For each $y \in Y$, since $f^{-1}(y)$ is Lindelöf, there is a countable subset $\{U_{yi}\}_{i\in\mathbb{N}}$ of $\mathscr{U}$ such that $f^{-1}(y) \subseteq \bigcup_{i\in\mathbb{N}} U_{yi} = U_y$. Since f is a closed mapping, according to [27, Theorem 1.4.13], there is an open set W_y of Y such that $f^{-1}(y) \subseteq f^{-1}(W_y) \subseteq U_y$. Clearly, $\{W_y : y \in Y\}$ is an open cover of Y. Since Y is a weakly θ-refinable space, there is an open refinement $\mathscr{L} = \bigcup_{j\in\mathbb{N}} \mathscr{L}_j$ of $\{W_y : y \in Y\}$ such that for each $y \in Y$, there is $m \in \mathbb{N}$ satisfying $1 \leq \mathrm{ord}(\mathscr{L}_m, y) < \omega$, where $\mathrm{ord}(\mathscr{L}_m, y) = |\{L \in \mathscr{L}_m : y \in L\}|$. Let $\mathscr{L}_j = \{L_{j\gamma} : \gamma \in \Gamma_j\}$, and $\mathscr{T}_j = \{f^{-1}(L_{j\gamma}) : L_{j\gamma} \in \mathscr{L}_j\}$. Then $\mathscr{T} = \bigcup_{j\in\mathbb{N}} \mathscr{T}_j$ is an open cover of X. For each $L_{j\gamma}$, take $W_{y(j\gamma)}$ such that $L_{j\gamma} \subseteq W_{y(j\gamma)}$. Thus, $f^{-1}(L_{j\gamma}) \subseteq f^{-1}(W_{y(j\gamma)}) \subseteq U_{y(j\gamma)} = \bigcup_{i\in\mathbb{N}} U_{y(j\gamma)i}$. Put $\mathscr{F}_{ji} = \{f^{-1}(L_{j\gamma}) \cap U_{y(j\gamma)i} : \gamma \in \Gamma_j\}$. Then $\mathscr{F} = \bigcup_{j,i\in\mathbb{N}} \mathscr{F}_{ji}$ is an open refinement of $\mathscr{U}$. For every $x \in X$, set $f(x) = y$, then there is $m \in \mathbb{N}$ satisfying $1 \leq \mathrm{ord}(\mathscr{L}_m, y) < \omega$. Since $x \in f^{-1}(L_{j\gamma}) \cap U_{y(j\gamma)i}$ for some j, i, γ, it follows that $y \in L_{j\gamma}$. Therefore, $1 \leq \mathrm{ord}(\mathscr{F}_m, x) < \omega$. It implies $1 \leq \mathrm{ord}(\mathscr{F}_{mi}, x) < \omega$ for every $i \in \mathbb{N}$. Thus X is a weakly θ-refinable space. □

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