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The k_R-property of free Abelian topological groups and products of sequential fans

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A space X is called a k_R-space, if X is Tychonoff and the necessary and sufficient condition for a real-valued function f on X to be continuous is that the restriction of f to each compact subset is continuous. In this paper, we discuss the k_R-property of products of sequential fans and free Abelian topological groups by applying the κ-fan introduced by Banakh. In particular, we prove the following two results:

(1) The space $S_{\omega_1} \times S_{\omega_1}$ is not a k_R-space.
(2) The space $S_\omega \times S_{\omega_1}$ is a k_R-space if and only if $S_\omega \times S_{\omega_1}$ is a k-space if and only if $\mathfrak{b} > \omega_1$.

These results generalize some well-known results on sequential fans. Furthermore, we generalize some results of Yamada on the free Abelian topological groups by applying the above results. Finally, we pose some open questions about the k_R-spaces.

1. Introduction

A topological space X is called a *k-space* if every subset of X, whose intersection with every compact subset K in X is relatively open in K, is open in X. It is well-known that the k-property which generalizes metrizability has been studied intensively by topologists and analysts. A space X is called a *k_R-space*, if X is Tychonoff and the necessary and sufficient condition for a real-valued function f on X to be continuous is that the restriction of f to each compact subset is continuous. Clearly every Tychonoff k-space is a k_R-space.

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The converse is false. Indeed, for any uncountable non-measurable cardinal κ the power $\mathbb{R}^\kappa$ is a k_R-space but not a k-space, see [27, Theorem 5.6] and [16, Problem 7.J(b)]. Now, the k_R-property has been widely used in the study of topology, analysis and category, see [4,3,5,6,17,21,26].

The results of our research will be presented in two separate papers. In the current paper, we extend some well-known results on k-spaces to k_R-spaces by applying the κ-fan introduced by Banakh in [3], and then seek some applications in the study of free Abelian topological groups. In the subsequent paper [19], we study the k_R-property in free topological groups.

Let κ be an infinite cardinal. For each $\alpha \in \kappa$, let T_α be a sequence converging to $x_\alpha \notin T_\alpha$. Let $T = \bigoplus_{\alpha\in\kappa}(T_\alpha \cup \{x_\alpha\})$ be the topological sum of $\{T_\alpha \cup \{x_\alpha\} : \alpha \in \kappa\}$. Then $S_\kappa = \{x\} \cup \bigcup_{\alpha\in\kappa} T_\alpha$ is the quotient space obtained from T by identifying all the points $x_\alpha \in T$ to the point x. The space S_κ is called a *sequential fan*. Throughout this paper, for convenience we denote S_ω and S_{ω_1} by the following respectively:

$S_\omega = \{a_0\} \cup \{a(n,m) : n, m \in \omega\}$, where for each $n \in \omega$ the sequence $a(n,m) \to a_0$ as $m \to \infty$;
$S_{\omega_1} = \{\infty\} \cup \{x(\alpha,n) : n \in \omega, \alpha \in \omega_1\}$, where for each $\alpha \in \omega_1$ the sequence $x(\alpha,n) \to \infty$ as $n \to \infty$.

The paper is organized as follows. In Section 2, we introduce the necessary notation and terminology which are used for the rest of the paper. In Section 3, we investigate the k_R-property of products of sequential fans. First, we prove that $S_{\omega_1} \times S_{\omega_1}$ is not a k_R-space, which generalizes a well-known result of Gruenhage and Tanaka. Then we prove that $S_\omega \times S_{\omega_1}$ is a k_R-space if and only if $S_\omega \times S_{\omega_1}$ is a k-space if and only if $\mathfrak{b} > \omega_1$. Furthermore, we discuss the topological properties of some class of spaces with the k_R-property under the assumption of $\mathfrak{b} \leq \omega_1$. Section 4 is devoted to the study of the k_R-property of free Abelian topological groups. The main theorems in this section generalizes some results in [18] and [33]. In Section 5, we pose some questions about k_R-spaces.

2. Preliminaries

In this section, we introduce the necessary notation and terminology. Throughout this paper, all topological spaces are assumed to be Tychonoff, unless otherwise is explicitly stated. First of all, let $\mathbb{N}$ be the set of all positive integers and ω the first infinite ordinal. For a space X, we always denote the set of all the non-isolated points by $\mathrm{NI}(X)$. For undefined notation and terminology, the reader may refer to [2], [8], [11] and [20].

Let X be a topological space and $A \subseteq X$ be a subset of X. The *closure* of A in X is denoted by $\overline{A}$ and the *diagonal* of X is denoted by Δ_X. Moreover, A is called *bounded* if every continuous real-valued function f defined on X is bounded on A. The space X is called a *k-space* provided that a subset $C \subseteq X$ is closed in X if $C \cap K$ is closed in K for each compact subset K of X. If there exists a family of countably many compact subsets $\{K_n : n \in \mathbb{N}\}$ of X such that each subset F of X is closed in X provided that $F \cap K_n$ is closed in K_n for each $n \in \mathbb{N}$, then X is called a *k_ω-space*. A space X is called a k_R-space, if X is Tychonoff and the necessary and sufficient condition for a real-valued function f on X to be continuous is that the restriction of f on each compact subset is continuous. Note that every k_ω-space is a k-space and every Tychonoff k-space is a k_R-space. A subset P of X is called a *sequential neighborhood* of $x \in X$, if each sequence converging to x is eventually in P. A subset U of X is called *sequentially open* if U is a sequential neighborhood of each of its points. A subset F of X is called *sequentially closed* if $X \setminus F$ is sequentially open. The space X is called a *sequential space* if each sequentially open subset of X is open. The space X is said to be *Fréchet–Urysohn* if, for each $x \in \overline{A} \subset X$, there exists a sequence $\{x_n\}$ in A such that $\{x_n\}$ converges to x.

A space X is called an S_2-space (*Arens' space*) if

$$X = \{\infty\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{n,m} : m, n \in \omega\}$$

and the topology is defined as follows: Each $x_{n,m}$ is isolated; a basic neighborhood of x_n is $\{x_n\} \cup \{x_{n,m} : m > k\}$, where $k \in \omega$; a basic neighborhood of ∞ is

$$\{\infty\} \cup (\bigcup\{V_n : n > k\}) \text{ for some } k \in \omega,$$

where V_n is a neighborhood of x_n for each $n \in \omega$.

Definition 2.1. ([3]) Let X be a topological space.

- A subset U of X is called $\mathbb{R}$-*open* if for each point $x \in U$ there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(X \setminus U) \subset \{0\}$. It is obvious that each $\mathbb{R}$-open set is open. The converse is true for the open subsets of Tychonoff spaces.
- A subset U of X is called a *functional neighborhood* of a set $A \subset X$ if there is a continuous function $f : X \to [0, 1]$ such that $f(A) \subset \{1\}$ and $f(X \setminus U) \subset \{0\}$. If X is normal, then each neighborhood of a closed subset $A \subset X$ is functional.

Definition 2.2. Let λ be a cardinal. An indexed family $\{X_\alpha\}_{\alpha\in\lambda}$ of subsets of a topological space X is called

- *point-countable* if for any point $x \in X$ the set $\{\alpha \in \lambda : x \in X_\alpha\}$ is countable;
- *compact-countable* if for any compact subset K in X the set $\{\alpha \in \lambda : K \cap X_\alpha \neq \emptyset\}$ is countable;
- *locally finite* if any point $x \in X$ has a neighborhood $O_x \subset X$ such that the set $\{\alpha \in \lambda : O_x \cap X_\alpha \neq \emptyset\}$ is finite;
- *compact-finite* in X if for each compact subset $K \subset X$ the set $\{\alpha \in \lambda : K \cap X_\alpha \neq \emptyset\}$ is finite;
- *strongly compact-finite* [3] in X if each set X_α has an $\mathbb{R}$-open neighborhood $U_\alpha \subset X$ such that the family $\{U_\alpha\}_{\alpha\in\lambda}$ is compact-finite;
- *strictly compact-finite* [3] in X if each set X_α has a functional neighborhood $U_\alpha \subset X$ such that the family $\{U_\alpha\}_{\alpha\in\lambda}$ is compact-finite.

Definition 2.3. ([3]) Let X be a topological space and λ be a cardinal. An indexed family $\{F_\alpha\}_{\alpha\in\lambda}$ of subsets of a topological space X is called a *fan* (more precisely, a λ-fan) in X if this family is compact-finite but not locally finite in X. A fan $\{X_\alpha\}_{\alpha\in\lambda}$ is called *strong* (resp. *strict*) if each set F_α has a $\mathbb{R}$-open neighborhood (resp. functional neighborhood) $U_\alpha \subset X$ such that the family $\{U_\alpha\}_{\alpha\in\lambda}$ is compact-finite in X.

If all the sets F_α of a λ-fan $\{F_\alpha\}_{\alpha\in\lambda}$ belong to some fixed family $\mathscr{F}$ of subsets of X, then the fan will be called an $\mathscr{F}^\lambda$-*fan*. In particular, if each F_α is closed in X, then the fan will be called a Cld^λ-*fan*.

Clearly, we have the following implications:

$$\text{strict fan} \;\Rightarrow\; \text{strong fan} \;\Rightarrow\; \text{fan}.$$

Let $\mathscr{P}$ be a family of subsets of a space X. Then, $\mathscr{P}$ is called a k-*network* if for every compact subset K of X and an arbitrary open set U containing K in X there is a finite subfamily $\mathscr{P}' \subseteq \mathscr{P}$ such that $K \subseteq \bigcup \mathscr{P}' \subseteq U$. Recall that a space X is an $\aleph$-*space* (resp. $\aleph_0$-*space*) if X has a σ-locally finite (resp. countable) k-network. Recall that a space X is said to be *Lašnev* if it is the continuous closed image of some metric space. We list two well-known facts about the Lašnev spaces as follows.

Fact 1: A Lašnev space is metrizable if it contains no closed copy of S_ω, see [24].

Fact 2: A Lašnev space is an $\aleph$-space if it contains no closed copy of S_{ω_1}, see [9] and [14].

Definition 2.4. ([7]) A topological space X is a *stratifiable space* if for each open subset U in X, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets in X such that the following (a)–(c) hold.

(a) $\overline{U_n} \subset U$;
(b) $\bigcup_{n=1}^{\infty} U_n = U$;
(c) $U_n \subset V_n$ whenever $U \subset V$.

Clearly, each Lašnev space is stratifiable [11].

Definition 2.5. Let X be a Tychonoff space. An Abelian topological group $A(X)$ is called *the free Abelian topological group* over X if $A(X)$ satisfies the following conditions:

(i) there is a continuous mapping $i : X \to A(X)$ such that $i(X)$ algebraically generates $A(X)$;
(ii) if $f : X \to G$ is a continuous mapping to an Abelian topological group G, then there exists a continuous homomorphism $\overline{f} : A(X) \to G$ with $f = \overline{f} \circ i$.

Let X be a non-empty Tychonoff space. Throughout this paper, $-X = \{-x : x \in X\}$, which is just a copy of X. Let 0 be the neutral element of $A(X)$, that is, the empty word. The word g is called *reduced* if it does not contain any pair of symbols of x and $-x$. It follows that if the word g is reduced and non-empty, then it is different from the neutral element 0 of $A(X)$. Clearly, each element $g \in A(X)$ distinct from the neutral element can be uniquely written in the form $g = r_1x_1 + r_2x_2 + \cdots + r_nx_n$, where $n \geq 1$, $r_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in X$, and $x_i \neq x_j$ for $i \neq j$. In this case, the number $\sum_{i=1}^{n} |r_i|$ is said to be the *reduced length* of g (in particular, the neutral element has the reduced length 0), and the *support* of $g = r_1x_1 + r_2x_2 + \cdots + r_nx_n$ is defined as $\mathrm{supp}(g) = \{x_1, \cdots, x_n\}$. Given a subset K of $A(X)$, we put $\mathrm{supp}(K) = \bigcup_{g \in K} \mathrm{supp}(g)$. For every $n \in \mathbb{N}$, let

$$i_n : (X \oplus -X \oplus \{0\})^n \to A_n(X)$$

be the natural mapping defined by

$$i_n(x_1, x_2, ...x_n) = x_1 + x_2 + ... + x_n$$

for each $(x_1, x_2, ...x_n) \in (X \oplus -X \oplus \{0\})^n$.

Let X be a space. For every $n \in \mathbb{N}$, $A_n(X)$ denotes the subspace of $A(X)$ that consists of all the words of reduced length at most n with respect to the free basis X.

The reader may refer to [28] for undefined notation and terminology of free groups.

3. The k_R-property in sequential fans

In this section we discuss the k_R-property of products of sequential fans and generalize some well-known results. First, we recall a well-known theorem of Gruenhage and Tanaka as follows:

Theorem 3.1. *([2, Corollary 7.6.23]) The product $S_{\omega_1} \times S_{\omega_1}$ is not a k-space.*

Next we generalize this theorem and prove that the product $S_{\omega_1} \times S_{\omega_1}$ is not a k_R-space. First of all, we give an important lemma.

Lemma 3.2. *([3, Proposition 3.2.1]) A k_R-space X contains no strict Cld-fan.*

Now we can prove one of the main results.

Theorem 3.3. *The space $S_{\omega_1} \times S_{\omega_1}$ is not a k_R-space.*

Proof. By Lemma 3.2, it suffices to prove that $S_{\omega_1} \times S_{\omega_1}$ contains a strict Cld$^\omega$-fan. Next we construct a strict Cld$^\omega$-fan in $S_{\omega_1} \times S_{\omega_1}$.

It follows from [15, Theorem 20.2] that we can find two families $\mathscr{A} = \{A_\alpha : \alpha \in \omega_1\}$ and $\mathscr{B} = \{B_\alpha : \alpha \in \omega_1\}$ of infinite subsets of ω such that

(a) $A_\alpha \cap B_\beta$ is finite for all $\alpha, \beta < \omega_1$;
(b) for no $A \subset \omega$, all the sets $A_\alpha \setminus A$ and $B_\alpha \cap A$, $\alpha \in \omega_1$ are finite.

For each $n \in \mathbb{N}$, put

$$X_n = \{(x(\alpha, n), x(\beta, n)) : n \in A_\alpha \cap B_\beta, \alpha, \beta \in \omega_1\}.$$

It is obvious that $(\infty, \infty) \notin \bigcup_{n\in\mathbb{N}} X_n$. However, it follows from the proof of [2, Lemma 7.6.22] that $(\infty, \infty) \in \overline{\bigcup_{n\in\mathbb{N}} X_n}$ but not for any countable subset of $\bigcup_{n\in\mathbb{N}} X_n$, which implies that the family $\{X_n :\in \mathbb{N}\}$ is not locally finite in $S_{\omega_1} \times S_{\omega_1}$. Moreover, it is easy to see that each X_n is closed and discrete in $S_{\omega_1} \times S_{\omega_1}$. We claim that the family $\{X_n : n \in \mathbb{N}\}$ is compact-finite in $S_{\omega_1} \times S_{\omega_1}$. Indeed, let K be an any compact subset of $S_{\omega_1} \times S_{\omega_1}$. It is easy to see that there exists a finite subset $\{\alpha_i \in \omega_1 : i = 1, \cdots, m\}$ of ω_1 such that

$$K \cap \bigcup_{n\in\mathbb{N}} X_n \subset (\bigcup_{i=1}^{m} \{x(\alpha_i, n) : n \in \mathbb{N}\}) \times (\bigcup_{i=1}^{m} \{x(\alpha_i, n) : n \in \mathbb{N}\}).$$

Assume that K intersects infinitely many X_n. Then there exist $1 \le i, j \le m$ and an infinite subset $\{n_k : k \in \mathbb{N}\}$ in $\mathbb{N}$ such that

$$\{x(\alpha_i, n_k), x(\alpha_j, n_k) : k \in \mathbb{N}\} \subset K \cap (\bigcup_{n\in\mathbb{N}} X_n).$$

It is obvious that

$$(\infty, \infty) \in \overline{\{x(\alpha_i, n_k), x(\alpha_j, n_k) : k \in \mathbb{N}\}}.$$

However, since the point (∞, ∞) does not belong to the closure of any countable subset of $\bigcup_{n\in\mathbb{N}} X_n$ in $S_{\omega_1} \times S_{\omega_1}$, we obtain a contradiction.

Since X_n is also a functional neighborhood of itself for each $n \in \mathbb{N}$, the space $S_{\omega_1} \times S_{\omega_1}$ contains a strict Cld$^\omega$-fan $\{X_n : n \in \mathbb{N}\}$. □

Next we prove the second main result in this section. First, we recall some concepts.

Consider ${}^\omega\omega$, the collection of all functions from ω to ω. For any $f, g \in {}^\omega\omega$, define $f \le g$ if $f(n) \le g(n)$ for all but finitely many $n \in \omega$. A subset $\mathscr{H}$ of ${}^\omega\omega$ is *bounded* if there is a $g \in {}^\omega\omega$ such that $f \le g$ for all $f \in \mathscr{H}$, and is *unbounded* otherwise. We denote by $\mathfrak{b}$ the smallest cardinality of an unbounded family in ${}^\omega\omega$. It is well-known that $\omega < \mathfrak{b} \le \mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of the continuum. Let $\mathscr{F}$ be the set of all finite subsets of ω.

In [10], Gruenhage proved that $\omega_1 < \mathfrak{b}$ if and only if the product $S_\omega \times S_{\omega_1}$ is a k-space. Indeed, we have the following result.

Theorem 3.4. *The following statements are equivalent.*

(1) $\mathfrak{b} > \omega_1$.
(2) *The product* $S_\omega \times S_{\omega_1}$ *is a* k_R*-space.*
(3) *The product* $S_\omega \times S_{\omega_1}$ *is a* k*-space.*

Proof. By Gruenhage's result, it suffices to prove (2) ⇒ (1). Assume that $\mathfrak{b} \leq \omega_1$. Then there exists a subfamily $\{f_\alpha \in {}^\omega\omega : \alpha < \omega_1\}$ such that for any $g \in {}^\omega\omega$ there exists $\alpha < \omega_1$ such that $f_\alpha(n) > g(n)$ for infinitely many n's. For each $\alpha < \omega_1$, put

$$G_\alpha = \{(a(n,m), x(\alpha,m)) : m \leq f_\alpha(n), n \in \omega\}.$$

Obviously, each G_α is clopen in $S_\omega \times S_{\omega_1}$. Moreover, it is easy to see that the family $\{G_\alpha\}_{\alpha<\omega_1}$ of subsets is compact-finite. However, the family $\{G_\alpha\}_{\alpha<\omega_1}$ is not locally finite at the point (a_0, ∞) in $S_\omega \times S_{\omega_1}$. Indeed, since each G_α is clopen in $S_\omega \times S_{\omega_1}$ and $(a_0, \infty) \notin G_\alpha$, it suffices to prove that $(a_0, \infty) \in \overline{\bigcup_{\alpha<\omega_1} G_\alpha}$. Take an arbitrary open neighborhood U at (a_0, ∞) in $S_\omega \times S_{\omega_1}$. Then there exists $f \in {}^\omega\omega$ and $g \in {}^{\omega_1}\omega$ such that

$$(\{a_0\} \cup \{a(n,m) : m \geq f(n), n \in \omega\}) \times (\{\infty\} \cup \{x(\alpha,m) : m \geq g(\alpha), \alpha < \omega_1\}) \subset U.$$

Therefore, there exists $\alpha < \omega_1$ such that $f_\alpha(n) > f(n)$ for infinitely many n's, then there is a $j \in \omega$ such that $j \geq g(\alpha)$ and $f_\alpha(j) > f(j)$, which shows $(a(j, f_\alpha(j)), x(\alpha, j)) \in U \cap G_\alpha$. By the arbitrary choice of U, the point (a_0, ∞) is in $\overline{\bigcup_{\alpha<\omega_1} G_\alpha}$.

Since $S_\omega \times S_{\omega_1}$ is normal and each G_α is clopen in $S_\omega \times S_{\omega_1}$, each G_α is also a functional neighborhood of itself. Therefore, it follows from Lemma 3.2 that $S_\omega \times S_{\omega_1}$ is not a k_R-space, which is a contradiction. Hence $\mathfrak{b} > \omega_1$. □

Note 3.5. If $\mathfrak{b} \leq \omega_1$, then some classes of spaces with the k_R-property have some special topological properties, see the following Theorem 3.9. In order to prove this theorem, we need some concepts and technique lemmas. Yamada in [33] introduced the following spaces.

Let $\mathfrak{T}$ be a class of metrizable spaces such that each element P of $\mathfrak{T}$ can be represented as $P = X_0 \cup \bigcup_{i=1}^\infty X_i$ satisfying the following conditions:

(1) X_i is an infinite discrete open subspace of P for every $i \in \mathbb{N}$,
(2) the family $\{X_i : i \in \omega\}$ is pairwise disjoint, and
(3) X_0 is a compact subspace of P, and $\{V_k = X_0 \cup \bigcup_{i=k}^\infty X_i : k \in \mathbb{N}\}$ is a neighborhood base at X_0 in P.

In the above definition, if each X_i consists of countably many elements and X_0 is a one-point set, we denote the space by P_0. Indeed, the space P_0 is commonly known as the metrizable countable hedgehog. We put $C = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with the subspace topology of I. Let

$$\mathbf{G}_0 = \bigoplus\{C_i : i \in \mathbb{N}\} \bigoplus P_0,$$

where each C_i is a copy of C for each $i \in \mathbb{N}$. Let

$$\mathbf{G}_1 = \bigoplus\{C_\alpha : \alpha < \omega_1\},$$

where $C_\alpha = \{c(\alpha, n) : n \in \mathbb{N}\} \cup \{c_\alpha\}$ with $c(\alpha, n) \to c_\alpha$ as $n \to \infty$ for each $\alpha \in \omega_1$.

From here on, we shall use the notations $\mathfrak{T}$, P_0, $\mathbf{G}_0$ and $\mathbf{G}_1$ with the meaning as the above meaning.

Lemma 3.6. *The space $S_\omega \times P$ is not a k_R-space for each space P in $\mathfrak{T}$.*

Proof. Fix an arbitrary space P in $\mathfrak{T}$. Then P can be represented as $X_0 \cup \bigcup_{i=1}^{\infty} X_i$ satisfying (1)–(3) in the above definition. Obviously, $S_\omega \times P$ is a normal space. By Lemma 3.2, it suffices to prove that $S_\omega \times P$ contains a strict closed$^\omega$-fan.

For each $n \in \mathbb{N}$, take an arbitrary countably infinite and pairwise disjoint subset $\{b(n,m) : m \in \mathbb{N}\}$ of X_n. For each $n \in \mathbb{N}$, put

$$F_n = \{(a(n,m), b(n,m)) : m \in \mathbb{N}\}.$$

Obviously, each F_n is closed. We claim that the family $\{F_n\}$ is compact-finite in $S_\omega \times P$. Indeed, for each compact subset K in $S_\omega \times P$, there exist compact subsets K_1 and K_2 in S_ω and P respectively such that $K \subset K_1 \times K_2$. Since K_1 is compact in S_ω, there exists a natural number n_0 such that

$$K_1 \subset \{a(n,m) : n \leq n_0, m \in \omega\} \cup \{a_0\}.$$

Therefore, $K \cap F_n = \emptyset$ for each $n > n_0$.

It is easy to see that

$$\emptyset \neq \overline{\bigcup_{n\in\mathbb{N}} F_n} \setminus \bigcup_{n\in\mathbb{N}} F_n \subset \{a_0\} \times X_0.$$

Hence the family $\{F_n\}$ is not locally finite at $S_\omega \times P$. Since $S_\omega \times P$ is a normal $\aleph$-space, it follows from [3, Proposition 2.9.2] that the family $\{F_n\}$ is strongly compact-finite, and then by the normality the family $\{F_n\}$ is also strictly compact-finite.

Therefore, the family $\{F_n\}$ is a strict closed$^\omega$-fan in $S_\omega \times P$. □

Proposition 3.7. *Let $S_\omega \times Z$ be a k_R-space, where Z is a stratifiable space. If $\{Z_n\}$ is a decreasing network for some point z_0 in Z, then the closure of Z_n is compact for some n.*

Proof. If not, then each $\overline{Z_n}$ is not compact and thus not countably compact. Hence there exist a sequence $\{n_k\}$ in $\mathbb{N}$ and the family $\{C_k\}$ of countably infinite, discrete and closed subsets of Z such that $C_k \subset \overline{Z_{n_k}} \setminus \{z_0\}$ for each $k \in \mathbb{N}$, and the family $\{C_k\}$ is pairwise disjoint. Put

$$Z_0 = \{z_0\} \cup \bigcup_{k\in\mathbb{N}} C_k.$$

It is easy to see that Z_0 belongs to $\mathfrak{T}$ and is closed in Z. Since Z is stratifiable, $S_\omega \times Z$ is stratifiable, hence $S_\omega \times Z_0$ is stratifiable. By [4, Proposition 5.10], $S_\omega \times Z_0$ is a k_R-subspace by the assumption. However, it follows from Lemma 3.6 that $S_\omega \times Z_0$ is not a k_R-subspace, which is a contradiction. Therefore, there exists $n \in \mathbb{N}$ such that the closure of Z_n is compact. □

By Proposition 3.7, we have the following corollary.

Corollary 3.8. *Let X be a stratifiable space with a point-countable k-network. If $S_\omega \times X$ is a k_R-space, then X has a point-countable k-network consisting of sets with compact closures.*

Let X and Y be two topological spaces. We recall that the pair (X, Y) satisfies the *Tanaka's conditions* [31] if one of the following (1)–(3) hold:

(1) Both X and Y are first-countable;
(2) Both X and Y are k-spaces, and X or Y is locally compact[2];
(3) Both X and Y are locally k_ω-spaces.[3]

Theorem 3.9. *Assume $\mathfrak{b} \leq \omega_1$, then the following statements hold.*

(1) *If X is a stratifiable k-space with a point-countable k-network and $S_\omega \times X$ is a k_R-space, then X is a locally σ-compact space; in particular, if X has a compact-countable k-network then X is a locally k_ω-space.*
(2) *If X is a stratifiable k-space with a point-countable k-network and $S_{\omega_1} \times X$ is a k_R-space, then X is a locally compact space.*
(3) *If both X and Y are stratifiable k-spaces with a compact countable k-network, then $X \times Y$ is a k_R-space if and only if the pair (X, Y) satisfies the conditions of Tanaka.*

Proof. By Theorem 3.4, we see that $S_\omega \times S_{\omega_1}$ is not a k_R-space.

(1) Assume that X is a stratifiable k-space and $S_\omega \times X$ is a k_R-space. By Corollary 3.8, it follows that X has a point-countable k-network consisting of sets with compact closures. We claim that there is no closed copy H of X such that H is the inverse image of S_{ω_1} under some perfect mapping. Assume to the contrary that there exist a closed subspace H of X and a perfect mapping $f : H \to S_{\omega_1}$ from H onto S_{ω_1}. Then it is easy to see that the mapping $\mathrm{id}_{S_\omega} \times f : S_\omega \times H \to S_\omega \times S_{\omega_1}$ is also an onto perfect mapping. Moreover, $S_\omega \times H$ is a k_R-space since $S_\omega \times X$ is a stratifiable k_R-space. Since the k_R-property is preserved by the perfect mappings, $S_\omega \times S_{\omega_1}$ is a k_R-space, which is a contradiction.

Since X is a k-space with a point-countable k-network consisting of sets with compact closures, it follows from [23, Lemma 1.3] that X contains no closed copy subspace of X such that it is the inverse image of S_{ω_1} under some perfect mapping if and only if X is a locally σ-compact space, thus X is a locally σ-compact space.

If X has a compact-countable k-network, then X is a locally $\aleph_0$-space. Then X is a locally k_ω-space since X is a k-space.

(2) Assume that X is a stratifiable k-space and $S_{\omega_1} \times X$ is a k_R-space. Moreover, since $S_\omega \times S_{\omega_1}$ is not a k_R-space, it is easy to see that X contain no closed copies of S_2 and S_ω. Then X is first-countable by [22, Corollary 3.9] since X is a k-space with a point-countable k-network. Since $S_\omega \times X$ is a closed subspace of the stratifiable space $S_{\omega_1} \times X$, the subspace $S_\omega \times X$ is a k_R-space. By Corollary 3.8, it follows that X has a point-countable k-network consisting of sets with compact closures. Then it follows from [25, Lemma 2.1] and the first-countability of X that X is locally compact.

(3) Assume that both X and Y are stratifiable k-spaces with a compact-countable k-network. Obviously, it suffices to prove that if $X \times Y$ is a k_R-space then (X, Y) satisfies the conditions of Tanaka. We divide the proof into the following cases.

Case 1: Both X and Y contain no closed copies of S_2 and S_ω.

Since both X and Y are k-spaces, it follows from [22, Corollary 3.9] that X and Y are all first-countable. Therefore, the pair (X, Y) satisfies the conditions of Tanaka.

Case 2: Both X and Y contain closed copies of S_2 or S_ω.

Then both $S_\omega \times Y$ and $X \times S_\omega$ are locally k_ω-spaces by (1). Therefore, the pair (X, Y) satisfies the conditions of Tanaka.

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[2] A space X is called *locally compact* if every point of X has a compact neighborhood.
[3] A space X is called *locally k_ω* if every point of X has a k_ω-neighborhood.

Case 3: The space X contains a closed copy of S_2 or S_ω and Y contains no closed copies of S_2 and S_ω.

Then Y is first-countable by [22, Corollary 3.9] and $S_\omega \times Y$ is a k_R-space since $X \times Y$ is stratifiable. By Corollary 3.8, it follows that Y has a compact-countable k-network consisting of sets with compact closures. Then it follows from [25, Lemma 2.1] and the first-countability of Y that Y is locally compact. Therefore, the pair (X, Y) satisfies the conditions of Tanaka.

Case 4: The space Y contains a closed copy of S_2 or S_ω and X contains no closed copies of S_2 and S_ω.

The proof is the same as the proof of Case 3. □

Corollary 3.10. *Assume $\mathfrak{b} \leq \omega_1$, then the following statements hold.*

(1) *If X is a Lašnev space and $S_\omega \times X$ is a k_R-space, then X is a locally k_ω-space.*
(2) *If X is a Lašnev space and $S_{\omega_1} \times X$ is a k_R-space, then X is a locally compact space.*
(3) *If X and Y are Lašnev spaces, then $X \times Y$ is a k_R-space if and only if the pair (X, Y) satisfies the conditions of Tanaka.*

Theorem 3.11. *Let X be a stratifiable space such that X^2 is a k_R-space. If X satisfies one of the following conditions, then either X is metrizable or X is the topological sum of k_ω-subspaces.*

(1) *X is a k-space with a compact-countable k-network;*
(2) *X is a Fréchet–Urysohn space with a point-countable k-network.*

Proof. First, suppose that X is a k-space with a compact-countable k-network. We divide the proof into the following three cases.

Case 1.1: The space X contains a closed copy of S_ω.

Since $S_\omega \times X$ is a closed subspace of the stratifiable space X^2, it follows from [4, Proposition 5.10] that $S_\omega \times X$ is a k_R-space. By Proposition 3.7, the space X has a compact-countable k-network consisting of sets with compact closures $\mathscr{P}$. Then $\mathscr{P}$ is star-countable, hence it follows from [13] that we have

$$\mathscr{P} = \bigcup_{\alpha \in A} \mathscr{P}_\alpha,$$

where each $\mathscr{P}_\alpha$ is countable and $(\bigcup \mathscr{P}_\alpha) \cap (\bigcup \mathscr{P}_\beta) = \varnothing$ for any $\alpha \neq \beta \in A$. For each $\alpha \in A$, put $X_\alpha = \bigcup \mathscr{P}_\alpha$. Since $\mathscr{P}$ is a k-network, it is easy to see that the family $\{X_\alpha : \alpha \in A\}$ is compact-finite in X. Put $\overline{\mathscr{P}} = \{\overline{P} : P \in \mathscr{P}\}$. We claim that $\overline{\mathscr{P}}$ is star-countable, hence $\overline{\mathscr{P}}$ is compact-countable since $\overline{\mathscr{P}}$ is a k-network in X.

Suppose not, there exists a $P \in \mathscr{P}$ and an uncountable subfamily $\{P_\alpha : \alpha < \omega_1\}$ of $\mathscr{P}$ such that $\overline{P} \cap \overline{P_\alpha} \neq \emptyset$ for each $\alpha < \omega_1$. Without loss of generality, we may assume that $P_\alpha \in \mathscr{P}_\alpha$ for each $\alpha < \omega_1$. Since each $\overline{P_\alpha}$ is metrizable, there exists a non-trivial sequence T_α in P_α converging to some point in $\overline{P}$. Without loss of generality, we may assume that $\overline{P} \cap T_\alpha = \emptyset$ for each $\alpha < \omega_1$. Let $F = \overline{P} \cup \bigcup_{\alpha < \omega_1} T_\alpha$. Since X is a k-space and the family $\{T_\alpha : \alpha < \omega_1\}$ is compact-finite, the set F is closed in X. Let $f : F \to F/P$ be the natural quotient mapping. Then f is perfect and F/P is homeomorphic to S_{ω_1}, hence F^2 is the inverse image of $(S_{\omega_1})^2$ under a perfect mapping. By Theorem 3.3, $(S_{\omega_1})^2$ is not a k_R-space, thus F^2 is not a k_R-space. However, since X is a stratifiable space and F is closed in X, the subspace F^2 is a k_R-space, which is a contradiction.

Therefore, without loss of generality, we may assume that $\mathscr{P}$ is a compact-countable compact k-network of X. Obviously, each X_α is a closed k-subspace of X and has a countable compact k-network $\mathscr{P}_\alpha$. Moreover, we claim that each X_α is open in X. Indeed, fix an arbitrary $\alpha \in A$. Since X is a k-space, it suffices to prove that $\bigcup\{X_\beta : \beta \in A, \beta \neq \alpha\} \cap K$ is closed in K for each compact subset K in X. Take an arbitrary compact subset K in X. Since $\mathscr{P}$ is a k-network of X, there exists a finite subfamily $\mathscr{P}' \subset \mathscr{P}$ such that $K \subset \bigcup \mathscr{P}'$. Then

$$\begin{aligned}\bigcup\{X_\beta : \beta \in A, \beta \neq \alpha\} \cap K &= \bigcup\{X_\beta : \beta \in A, \beta \neq \alpha\} \cap K \cap \bigcup \mathscr{P}' \\ &= K \cap \{P : P \in \mathscr{P}', P \notin \mathscr{P}_\alpha\}.\end{aligned}$$

Since each element of $\mathscr{P}'$ is compact, the set $\bigcup\{X_\beta : \beta \in A, \beta \neq \alpha\} \cap K$ is closed in K. Therefore, $X = \bigoplus_{\alpha \in A} X_\alpha$ and each X_α is a k_ω-subspace of X. Thus X is the topological sum of k_ω-subspaces.

Case 1.2: The space X contains a closed copy of S_2.

Obviously, $S_2 \times X$ is a k_R-space. Since S_ω is the image of S_2 under the perfect mapping and the k_R-property is preserved by the quotient mapping, $S_\omega \times X$ is a k_R-space. By Case 1.1, X is the topological sum of k_ω-subspaces.

Case 1.3: The space X contains no copy of S_ω or S_2.

Since X is a k-space with a point-countable k-network, it follows from [36, Lemma 8] and [22, Corollary 3.10] that X has a point-countable base, and thus X is metrizable since a stratifiable space with a point-countable base is metrizable [11].

Finally, let X be a Fréchet–Urysohn space with a point-countable k-network. We divide the proof into the following two cases

Case 2.1: The space X contains a closed copy of S_ω or S_2.

By Case 1.2, without loss of generality we may assume that X contains a closed copy of S_ω. By Proposition 3.7, the space X has a point-countable k-network consisting of sets with compact closures. Since X is a regular Fréchet–Urysohn space, it follows from [29, Corollary 3.6] that X is a Lašnev space, hence X has a compact-countable k-network. By Case 1.1, the space X is the topological sum of k_ω-subspaces.

Case 2.2: The space X contains no copy of S_ω or S_2.

It follows from [36, Lemma 8] and [22, Corollary 3.10] that X has a point-countable base. Then X is metrizable since a stratifiable space with a point-countable base is metrizable [11]. □

Remark 3.12. By the proof of Theorem 3.11, the condition (2) implies (1) in Theorem 3.11, and furthermore X^2 is a k-space.

4. The applications to free Abelian topological groups

In this section, we mainly discuss the k_R-property in the free Abelian topological groups. Recently, T. Banakh in [3] proved that $A(X)$ is a k-space if $A(X)$ is a k_R-space for a Lašnev space X. Indeed, he obtained this result in wider classes of spaces. However, he did not discuss the following question:

Question 4.1. *Let X be a space. For some $n \in \omega$, if $A_n(X)$ is a k_R-space, is $A_n(X)$ a k-space?*

We shall give some answers to the above question and generalize some results of Yamada in the free Abelian topological groups. First, we give a characterization for some class of spaces such that $A_2(X)$ is a k_R-space if and only if $A_2(X)$ is a k-space.

Theorem 4.2. *Let X be a stratifiable Fréchet–Urysohn space with a point-countable k-network. Then the following statements are equivalent:*

(1) *$A_2(X)$ is a k-space;*
(2) *$A_2(X)$ is a k_R-space;*
(3) *the space X is metrizable or X is a locally k_ω-space.*

Proof. Obviously, we have (1) ⇒ (2). It suffices to prove (3) ⇒ (1) and (2) ⇒ (3).

(3) ⇒ (1). Since $(X \cup \{0\} \cup (-X))^2$ is a k-space and i_2 is a closed mapping, $A_2(X)$ is a k-space.

(2) ⇒ (3). Use the same notations as in Theorem 3.3. First we claim that X contains no copy of S_{ω_1}. If not, let $Y_n = i_2(X_n)$ for each $n \in \mathbb{N}$. Since the family $\{X_n\}$ is a strict Cld$^\omega$-fan, it follows from [3, Proposition 3.4.3] that the family $\{Y_n\}$ is a strict Cld$^\omega$-fan, which implies that $A_2(X)$ is not a k_R-space, a contradiction. Therefore, X contains no closed copy of S_{ω_1}.

If X contains no close copy of S_ω, then it follows from [36, Lemma 8] and [22, Corollaries 3.9 and 3.10] that X has a point-countable base, thus it is metrizable since X is stratifiable [11]. Therefore, we may assume that X contains a closed copy of S_ω. Next we prove that X is a locally k_ω-space. Indeed we claim that X contains no closed subspace belonging to $\mathfrak{T}$. Assume to the contrary that X contains a closed subspace $P \in \mathfrak{T}$. By the proof of Lemma 3.6, $X \times X$ contains a closed Cld$^\omega$-fan. Since X is an $\aleph$-space, it follows from [3, Proposition 2.9.2] and the normality of $(X \cup \{0\} \cup (-X))^2$ that $(X \cup \{0\} \cup (-X))^2$ contains a strict Cld$^\omega$-fan. Since i_2 is a closed mapping, it follows from [3, Proposition 3.4.3] that $A_2(X)$ contains a strict Cld$^\omega$-fan, which is a contradiction. Hence X contains no closed subspace belonging to $\mathfrak{T}$, which means that every first-countable subspace is locally compact. Then it follows from [25, Lemma 2.1] that X has a point-countable k-network whose elements have compact closures. Finally, since X is a Fréchet–Urysohn space with a point-countable k-network, it follows from [23, Corollary 2.12] or [20, Corollary 5.4.10] that X is a locally k_ω-space if and only if X contains no closed copy of S_{ω_1}, thus X is a locally k_ω-space. □

Note 4.3. By Theorem 4.2, it follows that $A_2(S_{\omega_1})$ is not a k_R-space.

Lemma 4.4. *Let $A(X)$ be a k_R-space. If each $A_n(X)$ is a normal k-space, then $A(X)$ is k-space.*

Proof. It is well-known that each compact subset of $A(X)$ is contained in some $A_n(X)$ [2, Corollary 7.4.4]. Hence it follows from [17, Lemma 2] that $A(X)$ is a k-space. □

Theorem 4.5. *Let X be a paracompact σ-space.*[4] *Then $A(X)$ is a k_R-space and each $A_n(X)$ is a k-space if and only if $A(X)$ is a k-space.*

Proof. Since X is a paracompact σ-space, it follows from [2, Theorem 7.6.7] that $A(X)$ is also a paracompact σ-space, hence each $A_n(X)$ is normal. Now apply Lemma 4.4 to conclude the proof. □

In [33], Yamada proved that $A_4(\mathbf{G}_1)$ is not a k-space. Indeed, we prove that $A_4(\mathbf{G}_1)$ is not a k_R-space.

Suppose that $\mathscr{U}_X$ is the universal uniformity of a space X. Fix an arbitrary $n \in \mathbb{N}$. For each $U \in \mathscr{U}_X$ let

$$W_n(U) = \{x_1 - y_1 + x_2 - y_2 + \cdots + x_k - y_k : (x_i, y_i) \in U \text{ for } i = 1, \cdots, k, k \leq n\},$$

and $\mathscr{W}_n = \{W_n(U) : U \in \mathscr{U}_X\}$. Then the family $\mathscr{W}_n$ is a neighborhood base at 0 in $A_{2n}(X)$ for each $n \in \mathbb{N}$, see [33,34].

Proposition 4.6. *The subspace $A_4(\mathbf{G}_1)$ is not a k_R-space.*

[4] A regular space X is called a *σ-space* if it has a σ-locally finite network.

Proof. It suffices to prove that $A_4(\mathbf{G}_1)$ contains a strict Cld$^\omega$-fan. Let $\mathscr{A} = \{A_\alpha : \alpha \in \omega_1\}$ and $\mathscr{B} = \{B_\alpha : \alpha \in \omega_1\}$ be two families of infinite subsets of ω as in the proof of Theorem 3.3. For each $n \in \mathbb{N}$, put

$$X_n = \{c(\alpha, n) - c_\alpha + c(\beta, n) - c_\beta : n \in A_\alpha \cap B_\beta, \alpha, \beta \in \omega_1\}.$$

It suffices to prove the following three statements.

(1) The family $\{X_n\}$ is strictly compact-finite in $A_4(\mathbf{G}_1)$.

Since $\mathbf{G}_1$ is a Lašnev space, it follows from [2, Theorem 7.6.7] that $A(\mathbf{G}_1)$ is also a paracompact σ-space, hence $A_4(\mathbf{G}_1)$ is paracompact (and thus normal). Hence it suffices to prove that the family $\{X_n\}$ is strongly compact-finite in $A_4(\mathbf{G}_1)$. For each $\alpha \in \omega_1$ and $n \in \mathbb{N}$, let $C_\alpha^n = C_\alpha \setminus \{c(\alpha, m) : m \le n\}$, and put

$$U_n = \{c(\alpha, n) - x + c(\beta, n) - y : n \in A_\alpha \cap B_\beta, \alpha, \beta \in \omega_1, x \in C_\alpha^n, y \in C_\beta^n\}.$$

Obviously, each $X_n \subset A_4(\mathbf{G}_1) \setminus A_3(\mathbf{G}_1)$. Since $A_4(\mathbf{G}_1) \setminus A_3(\mathbf{G}_1)$ is open in $A_4(\mathbf{G}_1)$, it follows from [2, Corollary 7.1.19] that each U_n is open in $A_4(\mathbf{G}_1)$. We claim that the family $\{U_n\}$ is compact-finite in $A_4(\mathbf{G}_1)$. If not, then there exist a compact subset K in $A_4(\mathbf{G}_1)$ and a subsequence $\{n_k\}$ in $\mathbb{N}$ such that $K \cap U_{n_k} \neq \emptyset$ for each $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, choose an arbitrary point

$$z_k = c(\alpha_k, n_k) - x_k + c(\beta_k, n_k) - y_k \in K \cap U_{n_k},$$

where $x_k \in C_{\alpha_k}^{n_k}$ and $y_k \in C_{\beta_k}^{n_k}$. Since $A_4(\mathbf{G}_1)$ is paracompact, it follows from [1] that the closure of the set $\mathrm{supp}(K)$ is compact in $\mathbf{G}_1$. Therefore, there exists $N \in \mathbb{N}$ such that

$$\mathrm{supp}(K) \cap \bigcup\{C_\alpha : \alpha \in \omega_1 \setminus \{\gamma_i \in \omega_1 : i \le N\}\} = \emptyset,$$

that is, $\mathrm{supp}(K) \subset \bigcup_{\alpha \in \{\gamma_i \in \omega_1 : i \le N\}} C_\alpha$. Since each $z_k \in K$, there exists

$$\alpha_k, \beta_k \in \{\gamma_i \in \omega_1 : i \le N\}$$

such that $A_{\alpha_k} \cap B_{\beta_k}$ is an infinite set, which is a contradiction since $A_\alpha \cap B_\beta$ is finite for all $\alpha, \beta < \omega_1$.

(2) Each X_n is closed in $A_4(\mathbf{G}_1)$.

Fix an arbitrary $n \in \mathbb{N}$. Next we prove that X_n is closed in $A_4(\mathbf{G}_1)$. Let $Z = \mathrm{supp}(X_n)$. The set Z is a closed discrete subset of $\mathbf{G}_1$. Since $\mathbf{G}_1$ is metrizable, it follows from [32] that $A(Z)$ is topologically isomorphic to a closed subgroup of $A(\mathbf{G}_1)$, hence $A_4(Z)$ is a closed subspace of $A_4(\mathbf{G}_1)$. Since $A(Z)$ is discrete and $X_n \subset A_4(Z)$, the set X_n is closed in $A_4(Z)$ (and thus closed in $A_4(\mathbf{G}_1)$).

(3) The family $\{X_n\}$ is not locally finite at the point 0 in $A_4(\mathbf{G}_1)$.

Indeed, it suffices to prove that $0 \in \overline{\bigcup_{n\in\mathbb{N}} X_n} \setminus \bigcup_{n\in\mathbb{N}} X_n$. Take an arbitrary U that belongs to the universal uniformity on $\mathbf{G}_1$. We shall prove $W_2(U) \cap \bigcup_{n\in\mathbb{N}} X_n \neq \emptyset$.

Indeed, we can choose a function $f : \omega_1 \to \omega$ such that

$$V = \Delta_{\mathbf{G}_1} \cup \bigcup_{\alpha\in\omega_1} C_\alpha^{f(\alpha)} \times C_\alpha^{f(\alpha)} \subset U.$$

For each $\alpha < \omega_1$, put

$$A'_\alpha = \{n \in A_\alpha : n \ge f(\alpha)\}$$

and

$$B'_\alpha = \{n \in B_\alpha : n \geq f(\alpha)\}.$$

By the condition (b) of the families $\mathscr{A}$ and $\mathscr{B}$, it is easy to see that there exist $\alpha, \beta \in \omega_1$ such that $A'_\alpha \cap B'_\beta \neq \emptyset$. So, choose $n \in A'_\alpha \cap B'_\beta$. Then both $(c(\alpha, n), c_\alpha)$ and $(c(\beta, n), c_\beta)$ belong to V, thus

$$c(\alpha, n) - c_\alpha + c(\beta, n) - c_\beta \in W_2(V) \subset W_2(U),$$

which shows $W_2(U) \cap X_n \neq \emptyset$ (and thus $W_2(U) \cap \bigcup_{n\in\mathbb{N}} X_n \neq \emptyset$). □

Lemma 4.7. *Let X be a space. For each $n \in \mathbb{N}$, the subspace $A_{2^n-1}(X)$ of $A(X)$ contains a closed copy of X^n.*

Proof. Let the mapping $f : X^n \to A(X)$ defined by

$$f(x_1, \cdots, x_n) = x_1 + 2x_2 + \cdots + 2^{n-1}x_n$$

for each $(x_1, \cdots, x_n) \in X^n$. It follows from the proof of [2, Corollary 7.1.16] that f is a homeomorphic mapping from X^n onto $f(X^n)$. Then $A_{2^n-1}(X)$ contains a closed copy of X^n. □

Now we can prove the following one of the main results in this paper.

Theorem 4.8. *Let X be a non-metrizable stratifiable k-space with a compact-countable k-network. Then the following statements are equivalent:*

(1) *$A(X)$ is a sequential space;*
(2) *$A(X)$ is a k_R-space;*
(3) *each $A_n(X)$ is a k_R-space;*
(4) *$A_4(X)$ is a k_R-space;*
(5) *the space X is the topological sum of a k_ω-space and a discrete space.*

Proof. The implications of (1) ⇒ (2) and (3) ⇒ (4) are trivial. Since X is stratifiable, $A(X)$ is a stratifiable space by [30]. By [4, Proposition 5.10], the implication of (2) ⇒ (3) holds. It suffices to prove (5) ⇒ (1) and (4) ⇒ (5).

(5) ⇒ (1). Let $X = Y \bigoplus D$, where Y is a k_ω-space and D a discrete space. It is well-known that $A(X)$ is topologically isomorphic to $A(Y) \times A(D)$. Since $A(Y)$ is a k_ω-space by [2, Theorem 7.4.1] and $A(D)$ is a discrete space, it follows that $A(X)$ is a k-space (and thus a sequential space).

(4) ⇒ (5). First, we show the following claim.

Claim 1. *The subspace $NI(X)$ is ω_1-compact.*[5]

If not, then there exists a closed, discrete and uncountable subset $\{x_\alpha : \alpha < \omega_1\}$ in NI(X). Since X is paracompact and NI(X) is closed in X, there is an uncountable and discrete collection of open subsets $\{U_\alpha : \alpha < \omega_1\}$ in X such that $x_\alpha \in U_\alpha$ for each $\alpha < \omega_1$. For each $\alpha < \omega_1$, since X is sequential and $X \setminus \{x_\alpha\}$ is not sequentially closed, there exists a non-trivial sequence $\{x(n, \alpha) : n \in \mathbb{N}\}$ converging to x_α in X. For each $\alpha < \omega_1$, let

$$C_\alpha = \{x(n, \alpha) : n \in \mathbb{N}\} \cup \{x_\alpha\}$$

[5] Recall that a space is called ω_1-*compact* if every uncountable subset of X has a cluster point.

and put

$$Z = \bigcup\{C_\alpha : \alpha < \omega_1\}.$$

Obviously, Z is homeomorphic to $\mathbf{G}_1$. Without loss of generality, we may assume that $Z = \mathbf{G}_1$. Since $\mathbf{G}_1$ is a closed subset of X and X is a Lašnev space, it follows from [32] that the subspace $A_4(\mathbf{G}_1)$ is homeomorphic to a closed subset of $A_4(X)$, thus $A_4(\mathbf{G}_1)$ is a k_R-subspace. However, by Proposition 4.6, the subspace $A_4(\mathbf{G}_1)$ is not a k_R-subspace, which is a contradiction. Therefore, Claim 1 holds.

By the stratifiability of X, each compact subset of X is metrizable [11]. Then it follows from Theorem 3.11 and Lemma 4.7 that X is the topological sum of a family of k_ω-spaces. Let $X = \bigoplus_{\alpha\in A} X_\alpha$, where each X_α is a k_ω-space. Let

$$A' = \{\alpha \in A : X_\alpha \text{ is non-discrete}\}.$$

By Claim 1, the set A' is countable, hence X is the topological sum of a k_ω-space and a discrete space. □

Remark 4.9. The space X is a non-metrizable space in Theorem 4.8. It is natural to ask what happen when X is a metrizable space. Now we give an answer to this question, see Theorem 4.11 below. First we generalize a result of Yamada in [33], where Yamada proved that $A_3(\mathbf{G}_0)$ is not a k-space. Indeed, we prove that $A_3(\mathbf{G}_0)$ is not a k_R-space.

Proposition 4.10. *The subspace $A_3(\mathbf{G}_0)$ is not a k_R-space.*

Proof. Let $\mathbf{G}_0 = \bigoplus\{C_i : i \in \mathbb{N}\} \bigoplus P_0$, where $C_i = \{c(i,m) : m \in \mathbb{N}\} \cup \{c_i\}$ is a convergent sequence with the limit point c_i for each $i \in \mathbb{N}$, and let $\{y_0\} \cup \{y(n,m) : n,m \in \mathbb{N}\}$ be a closed copy of P_0 in $\mathbf{G}_0$, where the set $\{y(n,m) : m \in \mathbb{N}\}$ is discrete and open in $\mathbf{G}_0$ for each $n \in \mathbb{N}$. Since $A_3(\mathbf{G}_0)$ is a normal $\aleph_0$-space, it follows from [3, Proposition 2.9.2] that each compact-finite family is strongly compact-finite, hence by the normality the compact-finite family is also strictly compact-finite. Therefore, it suffices to prove that $A_3(\mathbf{G}_0)$ contains a Cld^ω-fan.

For each $n \in \mathbb{N}$, put

$$F_n = \{c_n - c(n,m) + y(n,m) : m \in \mathbb{N}\}.$$

We claim that the family $\{F_n\}$ is a Cld^ω-fan in $A_3(\mathbf{G}_0)$. We divide the proof into the following statements.

(1) Each F_n is closed in $A_3(\mathbf{G}_0)$.

Fix an arbitrary $n \in \mathbb{N}$. Indeed, let

$$X_n = \mathrm{supp}(F_n) = C_n \cup \{y(n,m) : m \in \mathbb{N}\}.$$

Clearly, X_n is closed in $\mathbf{G}_0$, thus $A(X_n)$ is topologically isomorphic to a closed subgroup of $A(\mathbf{G}_0)$. Hence $A_3(X_n)$ is closed in $A_3(\mathbf{G}_0)$. Since $F_n \subset A_3(X_n)$, it suffices to prove that F_n is closed in $A_3(X_n)$. By [35, Theorem 4.5], $A_3(X_n)$ is metrizable. Assume to the contrary that there exists a $g \in A_3(X_n)$ such that $g \in \overline{F_n}^{A_3(X_n)} \setminus F_n$, then there exists a sequence $\{c_n - c(n,m_k) + y(n,m_k)\}$ in F_n converging to g. Since X_n is paracompact, the closure of the set

$$\mathrm{supp}(\{g\} \cup \{c_n - c(n,m_k) + y(n,m_k) : k \in \mathbb{N}\})$$

is compact. However, the set

$$\mathrm{supp}(\{g\} \cup \{c_n - c(n, m_k) + y(n, m_k) : k \in \mathbb{N}\})$$

contains a closed infinite discrete subset $\{y(n, m_k) : k \in \mathbb{N}\}$, which is a contradiction. Therefore, F_n is closed in $A_3(X_n)$ (and thus closed in $A_3(\mathbf{G}_0)$).

(2) The family $\{F_n\}$ is compact-finite in $A_3(\mathbf{G}_0)$.

Assume to the contrary that there exist a compact subset K in $A_3(\mathbf{G}_0)$ and an increasing sequence $\{n_i : i \in \mathbb{N}\}$ in $\mathbb{N}$ such that $K \cap F_{n_i} \neq \emptyset$ for each $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, choose a point $c_{n_i} - c(n_i, m(i)) + y(n_i, m(i))$. Moreover, since $A_3(\mathbf{G}_0)$ is paracompact, the closure of the set $\mathrm{supp}(K)$ is compact. However, the set $\{c_{n_i} : i \in \mathbb{N}\} \subset \mathrm{supp}(K)$, which is a contradiction.

(3) The family $\{F_n\}$ is not locally finite at the point y_0 in $A_3(\mathbf{G}_0)$.

Clearly, it suffices to prove that $y_0 \in \overline{\bigcup_{n\in\mathbb{N}} F_n} \setminus \bigcup_{n\in\mathbb{N}} F_n$. Indeed, this was proved in [33, Theorem 3.4]. □

Theorem 4.11. *If X is a metrizable space, then the following statements are equivalent:*

(1) *$A_n(X)$ is a k-space for each $n \in \mathbb{N}$;*
(2) *$A_4(X)$ is a k-space;*
(3) *each $A_n(X)$ is a k_R-space;*
(4) *$A_4(X)$ is a k_R-space;*
(5) *either X is locally compact and the set $NI(X)$ is separable, or $NI(X)$ is compact.*

Proof. The equivalences of (1), (2) and (5) were proved in [33, Theorem 4.2]. Clearly, (1) ⇒ (3) and (3) ⇒ (4). It suffices to prove (4) ⇒ (5).

Assume that $A_4(X)$ is a k_R-space. By Claim 1 of the proof in Theorem 4.8, we see that $\mathrm{NI}(X)$ is separable. Assume that X is not locally compact and $\mathrm{NI}(X)$ is not compact. Then we can take an infinite discrete sequence $\{c_n \in \mathrm{NI}(X) : n \in \mathbb{N}\}$ in X. For each $n \in \mathbb{N}$, since $c_n \in \mathrm{NI}(X)$, there exists a convergent sequence $\{c(n, m) : m \in \mathbb{N}\}$ in X which converges to c_n, and put

$$C_n = \{c(n, m) : m \in \mathbb{N}\} \cup \{c_n\}.$$

Moreover, since X is not locally compact, there exists a closed copy of P_0 in X. Let

$$\{y_0\} \cup \{y(n, m) : n, m \in \mathbb{N}\}$$

be a closed copy of P_0 in X, where the set $\{y(n, m) : m \in \mathbb{N}\}$ is discrete and open in $\mathbf{G}_0$ for each $n \in \mathbb{N}$. Without loss of generality, we may assume that the collection

$$\{\{y_0\} \cup \{y(n, m) : n, m \in \mathbb{N}\}\} \cup \{C_n : n \in \mathbb{N}\}$$

is discrete in X. Let

$$Y = \{y_0\} \cup \{y(n, m) : n, m \in \mathbb{N}\} \cup \bigcup \{C_n : n \in \mathbb{N}\}.$$

Then Y is homeomorphic to $\mathbf{G}_0$. Hence, by Proposition 4.10, $A_3(\mathbf{G}_0)$ is not a k_R-space, which shows that $A_3(Y)$ is not a k_R-space. Since Y is closed in X and X is metrizable, $A_3(Y)$ is embedded into $A_3(X)$ as a closed subspace. Since $A(X)$ is stratifiable, it follows from [4, Proposition 5.10] that $A_3(X)$ is not a k_R-space. Then, by the same fact, $A_4(X)$ is not a k_R-space, which is a contradiction. □

By the proof of Theorem 4.11, we have the following theorem.

Theorem 4.12. *If X is a metrizable space, then the following statements are equivalent:*

(1) $A_3(X)$ *is a k-space;*
(2) $A_3(X)$ *is a k_R-space;*
(3) *either X is locally compact or $NI(X)$ is compact.*

Proof. The equivalence of (1) and (3) was proved in [33, Theorem 4.9], and (1) ⇒ (2) is obvious. The proof of Theorem 4.11 implies (2) ⇒ (3). □

For each space X, the subspace $A_3(X)$ contains a closed copy of $X \times X$ by Lemma 4.7, hence it follows from Theorems 3.11 and 4.12 that we have the following corollary.

Corollary 4.13. *Let X be a stratifiable k-space with a compact-countable k-network. If $A_3(X)$ is a k_R-space, then X satisfies one of the following* (a)–(c)*:*

(a) *X is a locally compact metrizable space;*
(b) *X is a metrizable space with the set of all non-isolated points being compact;*
(c) *X is the topological sum of k_ω-subspaces.*

Theorem 4.14. *Assume $\mathfrak{b} = \omega_1$. Let X be a stratifiable k-space with a compact-countable k-network. If $A_3(X)$ is a k_R-space, then $A_3(X)$ is a k-space.*

Proof. Since $A_3(X)$ is a k_R-space, it follows from Lemma 4.7 that X^2 is a k_R-space. By Theorem 3.11, either X is metrizable or X is the topological sum of k_ω-subspaces. If X is a metrizable space, then it follows from Theorem 4.12 that $A_3(X)$ is a k-space. Now we may assume that X is a non-metrizable space being the topological sum of k_ω-subspaces. Moreover, by the assumption of $\mathfrak{b} = \omega_1$, there exists a collection $\{f_\alpha \in {}^\omega\omega : \alpha < \omega_1\}$ such that if $f \in {}^\omega\omega$, then there exists $\alpha < \omega_1$ with $f_\alpha(n) > f(n)$ for infinitely may $n \in \omega$. Now we shall prove that $A_3(X)$ is a k-space.

Indeed, it suffices to prove that $\mathrm{NI}(X)$ is ω_1-compact. Assume to the contrary that the subspace $\mathrm{NI}(X)$ is not ω_1-compact. Since X is sequential, we can see that X contains a closed copy of $\mathbf{G}_1 = \bigoplus\{C_\alpha : \alpha < \omega_1\}$, where for each $\alpha \in \omega_1$ the set

$$C_\alpha = \{c(\alpha, n) : n \in \omega\} \cup \{c_\alpha\}$$

and $c(\alpha, n) \to c_\alpha$ as $n \to \infty$. Next we divide the proof into the following two cases.

Case 1: The space X contains no closed copy of S_ω.

If X contains no closed copy of S_2. Then it follows from [22, Corollaries 2.13 and 3.10] and [36, Lemma 8] that X has a point-countable base, thus X is metrizable since a stratifiable space with a point-countable base is metrizable [11], which is a contradiction. Therefore, we may assume that X contains a closed copy of S_2. Put

$$X_1 = \{\infty\} \cup \bigcup_{n\in\omega} D_n,$$

where $D_n = \{x_n\} \cup \{x_n(m) : m \in \omega\}$ for each $n \in \omega$. For each $n, k \in \omega$, put

$$D_n^k = \{x_n\} \cup \{x_n(m) : m > k\}.$$

We endow X_1 with a topology as follows: each $x_n(m)$ is isolated; the family $\{D_n^k\}$ is a neighborhood base at the point x_n for each $n \in \omega$; a basic neighborhood of ∞ is

$$N(f,F)=\{\infty\}\cup\bigcup\{D_n^{f(n)}:n\in\omega-F\},$$

where $f\in{}^{\omega}\omega$ and $F\in\mathscr{F}$. Then X_1 is a closed copy of S_2. Moreover, without loss of generality, we may assume that $X_1\subset X$ and $X_1\cap\mathbf{G}_1=\emptyset$. Let $X_2=X_1\cup\mathbf{G}_1$.

For arbitrary $n,m\in\omega$ and $\beta\in\omega_1$, let

$$F_{n,m}=\{x_n(m)+c(\alpha,n)-c_\alpha:m\le f_\alpha(n),\alpha\in\omega_1\},$$
$$V_{n,m}=\{x_n(m)+c(\alpha,n)-x:m\le f_\alpha(n),\alpha\in\omega_1,x\in C_\alpha^n\},$$

and

$$C_\beta^n=C_\beta\setminus\{c(\beta,k):k\le n\}.$$

Since X is stratifiable, it follows from [30] that $A(X)$ is stratifiable. Then it follows from [4, Proposition 5.10] and [32] that $A_3(X_2)$ is a k_R-subspace. However, we shall claim that the family $\{F_{n,m}\}$ is a Cld^ω-fan in $A_3(X_2)$ and the family $\{V_{n,m}\}$ is compact-finite in $A_3(X_2)$; then since $A_3(X_2)$ is normal, it follows that the family $\{F_{n,m}\}$ is a strict Cld^ω-fan in $A_3(X_2)$, which is a contradiction. We divide the proof into the following three statements.

(1) Each $F_{n,m}$ is closed in $A_3(X_2)$.

Fix arbitrary $n,m\in\omega$, and let $X_{n,m}=\mathrm{supp}(F_{n,m})$. Obviously, $X_{n,m}$ is a closed discrete subspace of X_2. By an argument similar to the proof of (2) in Proposition 4.6, $F_{n,m}$ is closed in $A_3(X_2)$.

(2) The family $\{V_{n,m}\}$ is compact-finite in $A_3(X_2)$.

Since the proof is similar to (1) in Proposition 4.6, we omit it.

(3) The family $\{F_{n,m}\}$ is not locally finite in $A_3(X_2)$.

It suffices to prove that the family $\{F_{n,m}\}$ is not locally finite at the point ∞ in $A_3(X_2)$. We give a uniform base $\mathcal{U}$ of the universal uniformity on X_2 as follows. For each $\alpha<\omega_1$ and $n,k\in\omega$, let

$$W_{k,n}=(D_n^k\times D_n^k)\cup\Delta_{x_n}\text{and } U_{k,\alpha}=(C_\alpha^k\times C_\alpha^k)\cup\Delta_\alpha,$$

where Δ_{x_n} and Δ_α are the diagonals of $D_n\times D_n$ and $C_\alpha\times C_\alpha$ respectively. For each $f\in{}^{\omega}\omega$, $g\in{}^{\omega_1}\omega$ and $F\in\mathscr{F}$, let

$$U(g,f,F)=\bigcup\{U_{g(\alpha),\alpha}:\alpha<\omega_1\}\cup(N(f,F)\times N(f,F))\cup(\bigcup_{n\in\omega}W_{f(n),n}\times W_{f(n),n})\cup\Delta_{X_2}.$$

Put

$$\mathscr{U}=\{U(g,f,F):g\in{}^{\omega_1}\omega,f\in{}^{\omega}\omega,F\in\mathscr{F}\}.$$

Then the family $\mathscr{U}$ is a uniform base of the universal uniformity on the space X_2. Put

$$\mathcal{W}=\{W(P):P\in\mathscr{U}^\omega\}.$$

Then it follows from [33] that $\mathcal{W}$ is a neighborhood base at 0 in $A(X_2)$.

Next we prove that $\infty\in\overline{H}\setminus H$ in $A_3(X_2)$, where $H=\bigcup_{n,m\in\omega}F_{n,m}$. Obviously, the family $\{(\infty+U)\cap A_3(X_2):U\in\mathcal{W}\}$ is a neighborhood base at ∞ in $A_3(X_2)$. We shall prove $(\infty+U)\cap A_3(X_2)\cap H\neq\emptyset$ for each $U\in\mathcal{W}$, which implies $\infty\in\overline{H}\setminus H$ in $A_3(X_2)$. Fix an $U\in\mathcal{W}$. Then there exist a sequence $\{h_i\}_{i\in\omega}$ in ${}^{\omega_1}\omega$, a sequence $\{g_i\}_{i\in\omega}$ in ${}^{\omega}\omega$ and a sequence $\{F_i\}_{i\in\omega}$ in $\mathscr{F}$ such that

$$U = \{x_1 - y_1 + x_2 - y_2 + ... + x_n - y_n : (x_i, y_i) \in U(h_i, g_i, F_i), i \leq n, n \in \omega\}.$$

Let

$$B = \{x' - \infty + x'' - y'' : (x', \infty) \in N(g_1, F_1) \times N(g_1, F_1), (x'', y'') \in U_{h_1(\alpha),\alpha}, \alpha < \omega_1\}.$$

Then $B \subset U$. By the assumption, there exists $\alpha < \omega_1$ such that $f_\alpha(k) \geq g_1(k)$ for infinitely many k. Pick a $k' > h_1(\alpha)$ such that $k' \notin F_1$ and $(x_{k'}(f_\alpha(k')), \infty) \in N(g_1, F_1) \times N(g_1, F_1)$. Then

$$x_{k'}(f_\alpha(k')) - \infty + c(\alpha, k') - c_\alpha \in U,$$

hence

$$\begin{aligned} \infty + x_{k'}(f_\alpha(k')) - \infty + c(\alpha, k') - c_\alpha &= x_{k'}(f_\alpha(k')) + c(\alpha, k') - c_\alpha \\ &\in ((\infty + U) \cap A_3(Z)) \cap F_{k', f_\alpha(k')} \\ &\subset F_{k', f_\alpha(k')}. \end{aligned}$$

Case 2: The space X contains a closed copy of S_ω.

Then X contains a closed subspace Y which is homeomorphic to S_ω. Moreover, without loss of generality, we may assume that $Y \cap \mathbf{G}_1 = \emptyset$. Let $Z = Y \cup \mathbf{G}_1$. For arbitrary $n, m \in \omega$, let

$$H_{n,m} = \{a(n, m) + c(\alpha, n) - c_\alpha : m \leq f_\alpha(n), \alpha \in \omega_1\}.$$

By an argument similar to the proof of Case 1, we can prove that the family $\{H_{n,m}\}$ is a strict Cld$^\omega$-fan in the k_R-subspace $A_3(Z)$, which is a contradiction. □

Note 4.15. By the above proof, it follows that if the space X in Theorem 4.14 is non-metrizable then $A(X)$ is a k-space.

5. Open questions

In this section, we pose some open questions about k_R-spaces.

In [12], the authors proved that each countably compact k-space with a point-countable k-network is compact and metrizable. Therefore, we have the following question.

Question 5.1. *Is each countably compact k_R-space with a point-countable k-network metrizable?*

In [4], the authors proved that each closed subspace of a stratifiable k_R-space is a k_R-subspace. However, the following question is still open. First, we recall a concept.

Definition 5.2. A topological space X is a *k-semistratifiable space* if for each open subset U in X, one can assign a sequence $\{U_n\}_{n=1}^\infty$ of closed subsets in X such that

(a) $U = \bigcup_{n \in \mathbb{N}} U_n$;
(b) for each compact subset K with $K \subset U$ there exists $n \in \mathbb{N}$ such that $K \subset U_n$;
(c) $U_n \subset V_n$ whenever $U \subset V$.

Question 5.3. *Is each closed subspace of k-semistratifiable k_R-space a k_R-subspace?*

Moreover, we have the following question:

Question 5.4. *Is each closed subgroup of a k_R-topological group (free Abelian topological group) k_R?*

In [3], the author posed the following question:

Problem 5.5. ([3, Problem 3.5.5]) Assume that a Tychonoff space contains no Cld-fan. Is X a k-space?

Indeed, the following question is still open.

Question 5.6. *Assume that a topological group G contains no Cld-fan. Is G a k-space?*

Furthermore we have the following question.

Question 5.7. *Assume that a topological group G contains no strict Cld-fan. Is G a k_R-space?*

By Theorem 3.9, we have the following question.

Question 5.8. *Can we replace "k-space" with "k_R-space" in the conditions (1), (2) and (3) in Theorem 3.9?*

We conjecture that the answer to the following question is positive.

Question 5.9. *Let X and Y be two sequential spaces. If $X \times Y$ is a k_R-space, is $X \times Y$ a k-space?*

By Theorem 4.14, it is natural to pose the following question.

Question 5.10. *Assume $\mathfrak{b} > \omega_1$. Let X be a stratifiable k-space with a compact-countable k-network. If $A_3(X)$ is a k_R-space, then is $A_3(X)$ a k-space?*

It is well-known that neither $(S_2)^\omega$ nor $(S_\omega)^\omega$ are k-spaces. Hence it is natural to pose the following question:

Question 5.11. *Are $(S_2)^\omega$ and $(S_\omega)^\omega$ k_R-spaces?*

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